CSE 250
Data Structures

Dr. Eric Mikida
epmikida@buffalo.edu
208 Capen Hall

Lec 05: Asymptotic Analysis

Announcements and Feedback

- Join Piazza! (Link on course website)
- Normal recitations (w/attendance) begin next week
- Academic Integrity Quiz due 2/4 @ 11:59PM **(MUST GET 100%)**
- PA0 due 2/4 @ 11:59PM **(MUST GET 100%)**
- WA1 due 2/4 @ 11:59PM

Analysis Checklist

1. Don't think in terms of wall-time, think in terms of "number of steps"
2. To give a useful solution, we should take "scale" into account
    - How does the runtime change as we change the size of the input?
3. Focus on "large" inputs
    - Rank functions based on how they behave at large scales
4. Decouple algorithm from infrastructure/implementation

Attempt #1: Wall-clock time?

- What is fast?
  - 10s? 100ms? 10ns?
  - …it depends on the task
- Algorithm vs Implementation
  - Compare Grace Hopper's implementation to yours
- What machine are you running on?
  - Your old laptop? A lab machine? The newest, shiniest processor?
- What bottlenecks exist? CPU vs IO vs Memory vs Network…

Wall-clock time is not terribly useful…

Attempt #2: Growth Functions

Not a function in code…but a mathematical function:

$$T(n)$$

n: The "size" of the input

ie: number of users,rows, pixels, etc

***T*(n): The number of "steps" taken for input of size n**

ie: 20 steps per user, where n = |Users|, is 20 x n

Some Basic Assumptions:

Problem sizes are non-negative integers

$$n \in \{0, 1, 2, 3, \ldots\} = \{0\} \cup \mathbb{Z}^+$$

We can't reverse time…(obviously)

$$T(n) > 0$$

Smaller problems aren't harder than bigger problems

$$n_1 < n_2 \Rightarrow T(n_1) \leq T(n_2)$$

Some Basic Assumptions:

Problem sizes are non-negative integers

$$n \in \{0, 1, 2, 3, \ldots\} = \{0\} \cup \mathbb{Z}^+$$

We can’t reverse time…(obviously)

$$T(n) > 0$$

Smaller problems aren’t harder than bigger problems

$$n_1 < n_2 \Rightarrow T(n_1) \leq T(n_2)$$

$T: \{0\} \cup \mathbb{Z}^+ \to \mathbb{R}^+$

T is non-decreasing

First Problem...

We are still implementation dependent...

$$T_1(n) = 19n$$

$$T_2(n) = 20n$$

First Problem...

We are still implementation dependent...

$$T_1(n) = 19n$$

$$T_2(n) = 20n$$

Does 1 extra step per element really matter...?

Is this just an implementation detail?

First Problem...

We are still implementation dependent...

$$T_1(n) = 19n$$

$$T_2(n) = 20n$$

$$T_3(n) = 2n^2$$

T_1 and T_2 are much more "similar" to each other than they are to T_3

First Problem...

We are still implementation dependent...

$$T_1(n) = 19n$$

$$T_2(n) = 20n$$

$$T_3(n) = 2n^2$$

T_1 and T_2 are much more "similar" to each other than they are to T_3

How do we capture this idea formally?

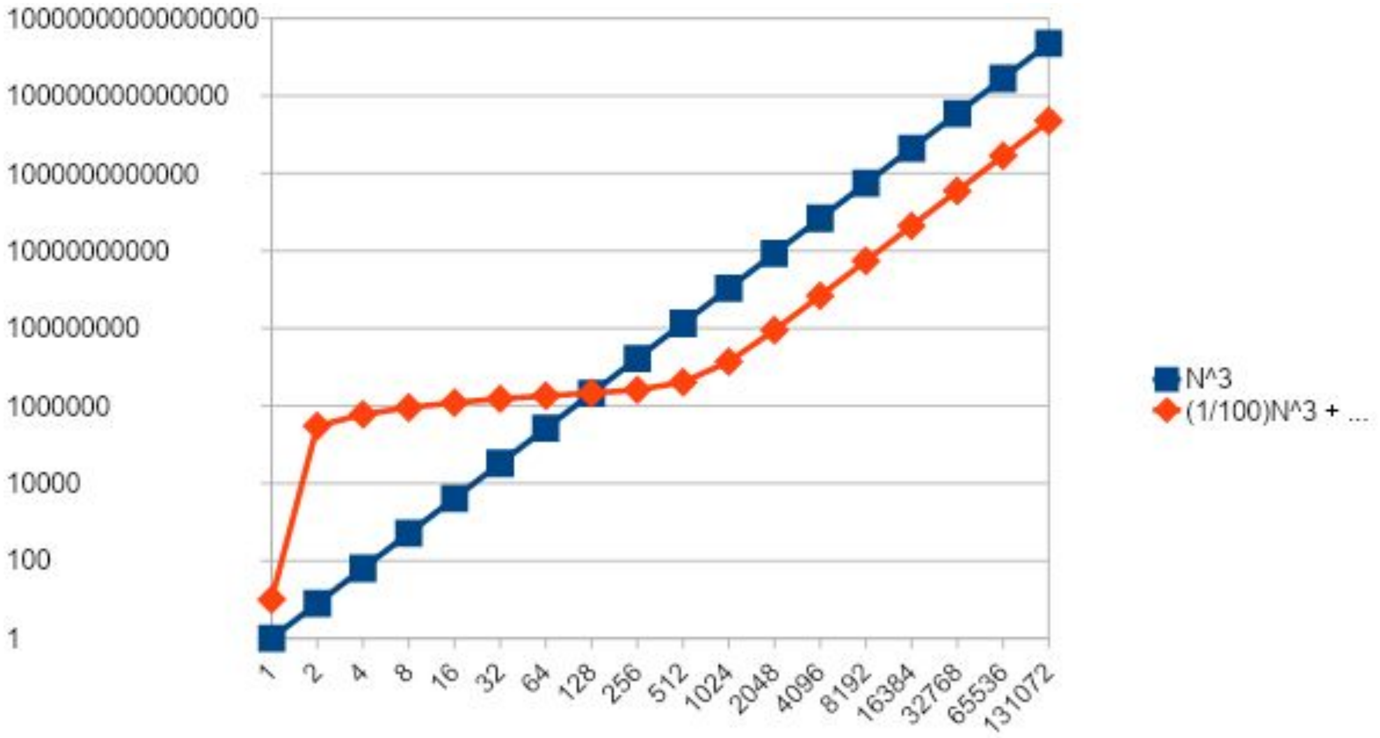
How Do We Capture Behavior at Scale?

Consider the following two functions:

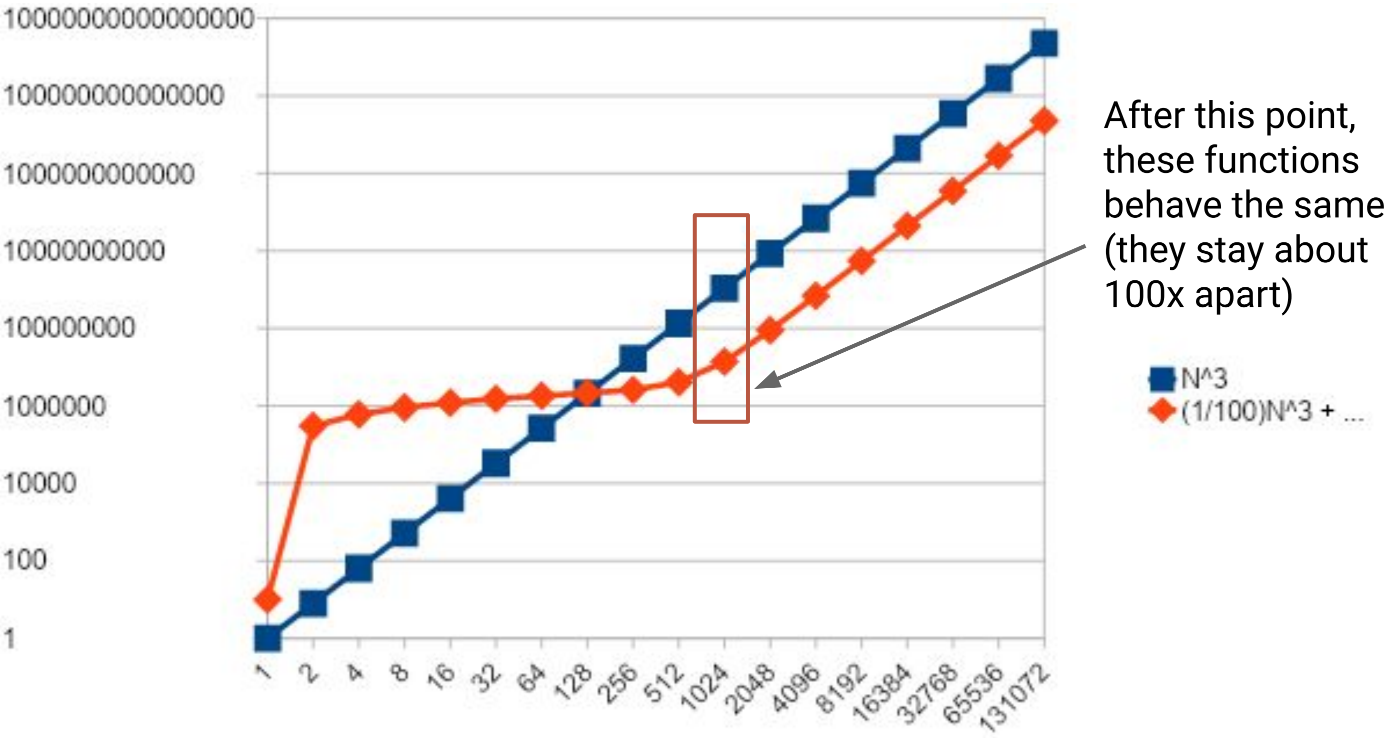
$$\frac{1}{100}n^3 + 10n + 1000000\log(n)$$

$$n^3$$

How Do We Capture Behavior at Scale?



How Do We Capture Behavior at Scale?



After this point, these functions behave the same (they stay about 100x apart)

Attempt #3: Asymptotic Analysis

We want to organize runtimes (growth functions) into different ***Complexity Classes***

Within the same complexity class, runtimes “behave the same”/"have the same shape" (at scale)

Getting More Formal

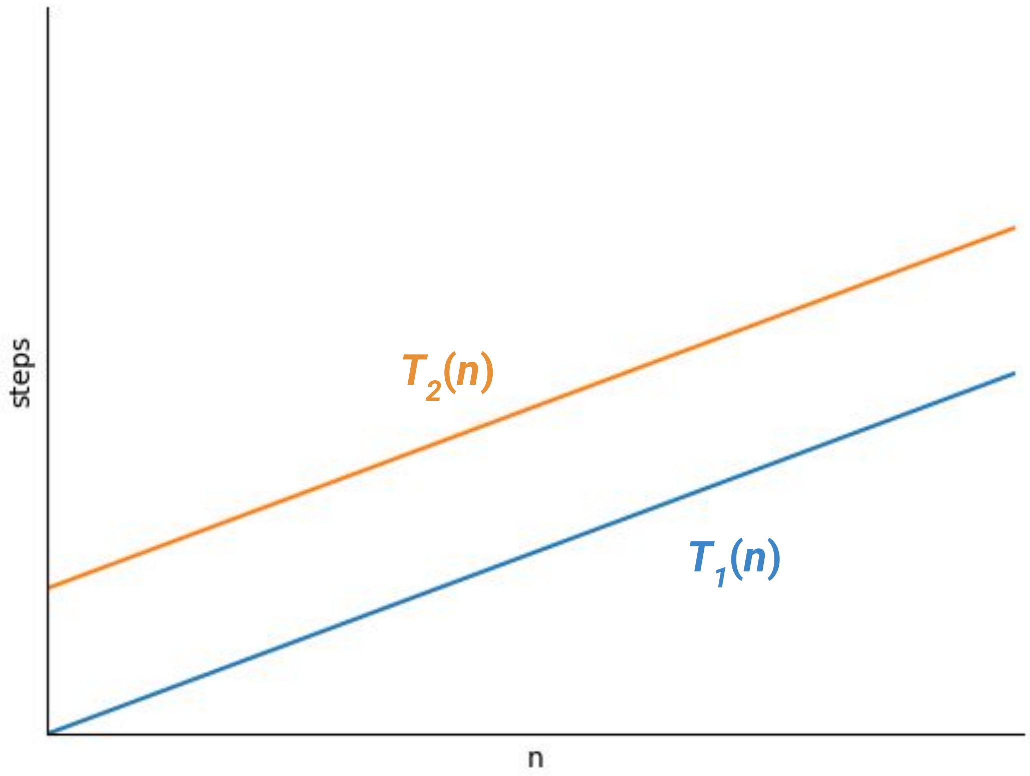
When do we consider two functions to have the same shape?

Additive Factors

Consider two growth functions:

$T_1(n) = 3n$

$T_2(n) = 3n + 3$



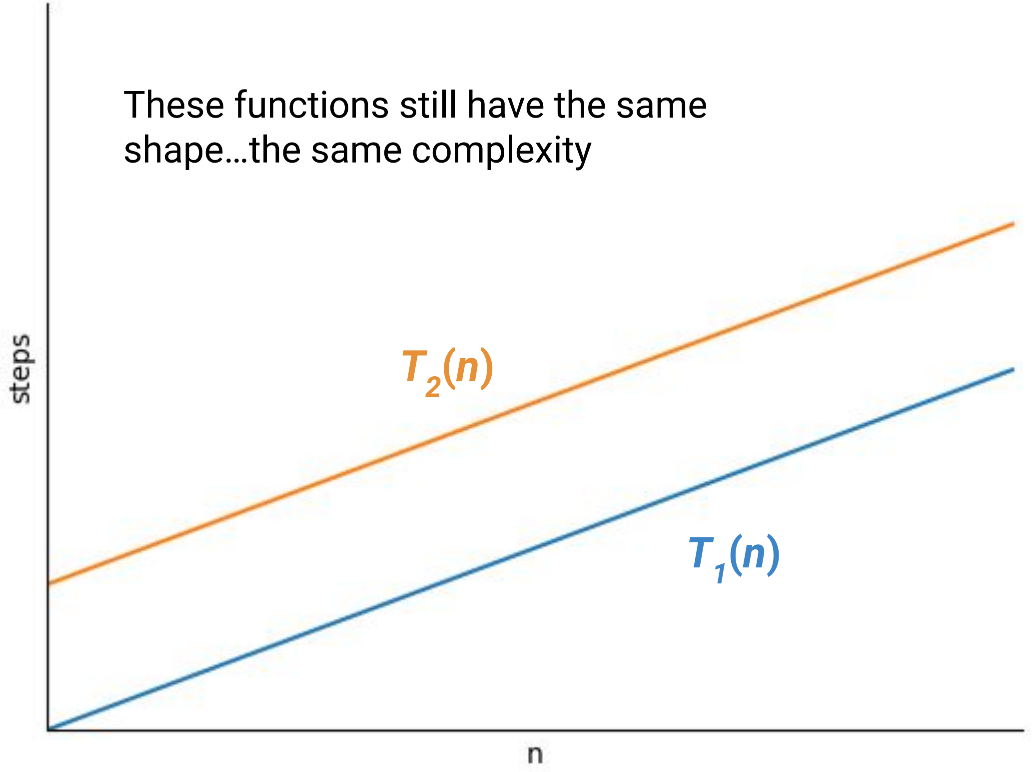
Additive Factors

Consider two growth functions:

$T_1(n) = 3n$

$T_2(n) = 3n + 3$

These functions still have the same shape…the same complexity

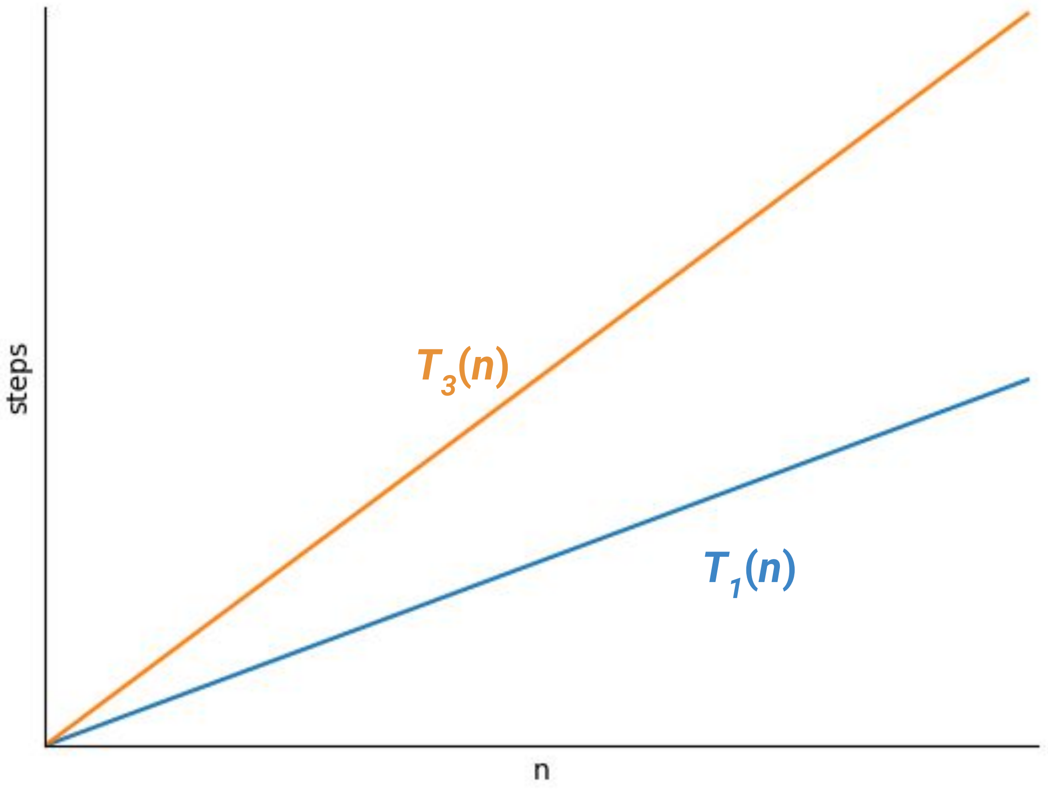


Multiplicative Factors

Consider two growth functions:

$T_1(n) = 3n$

$T_3(n) = 6n$

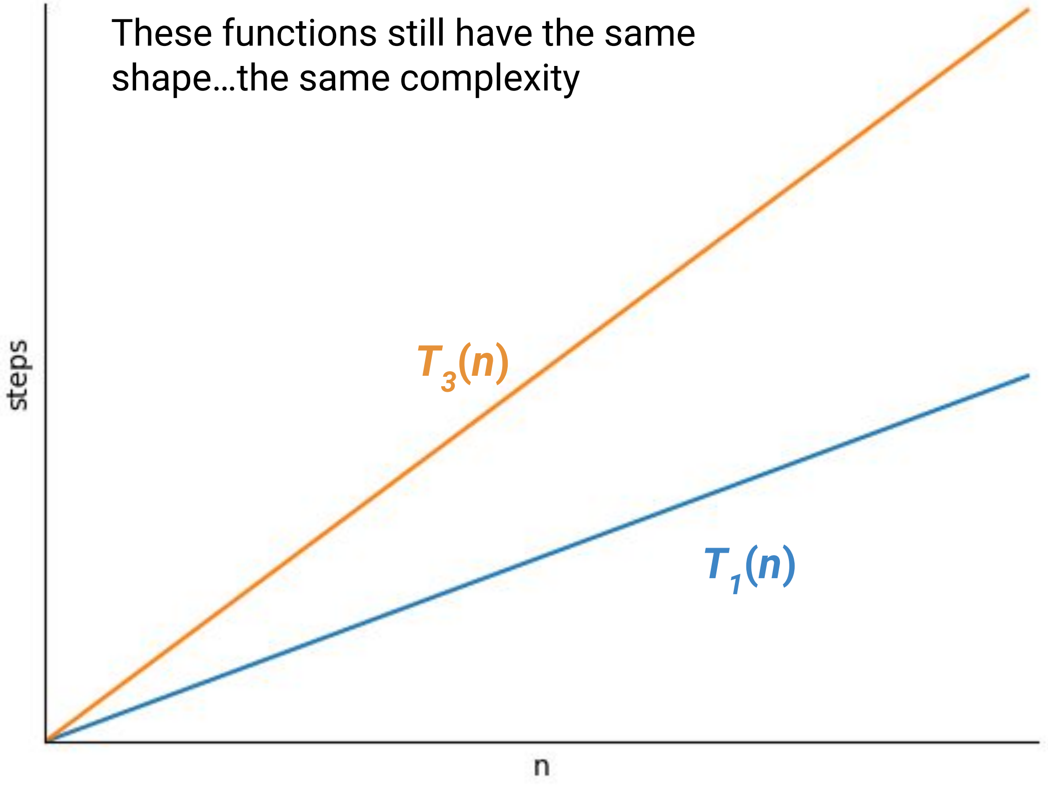


Multiplicative Factors

Consider two growth functions:

$T_1(n) = 3n$

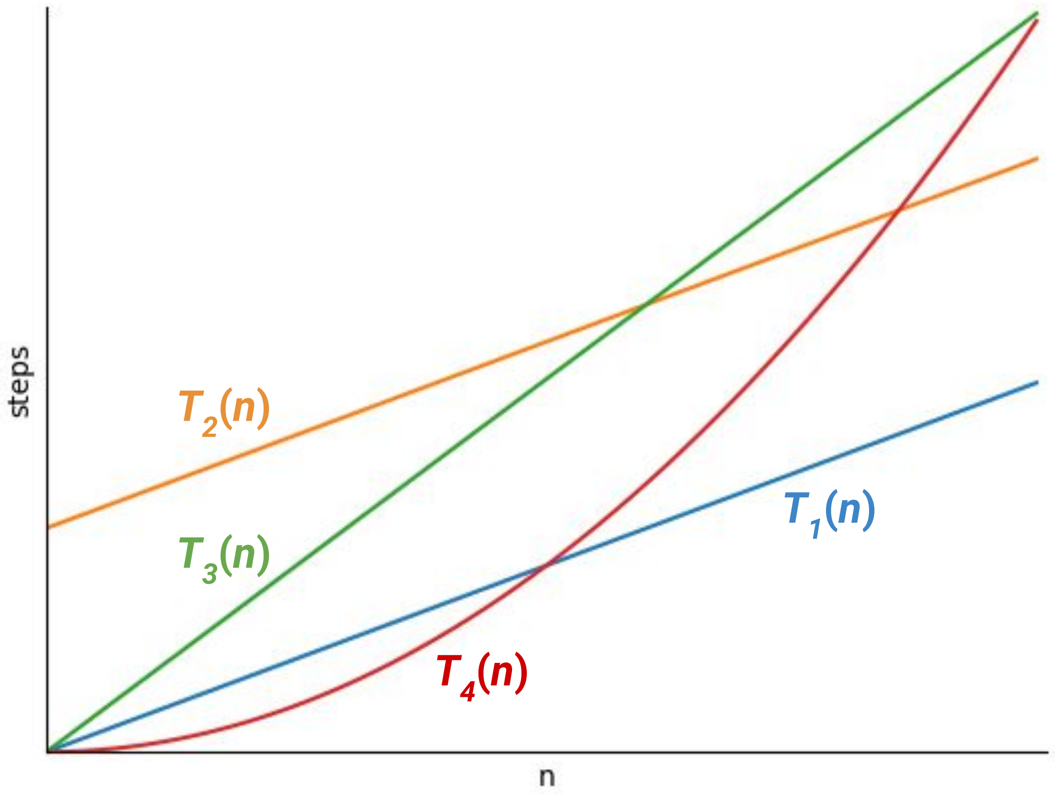
$T_3(n) = 6n$



A Counter Example

Now consider:

$T_4(n) = n^2$

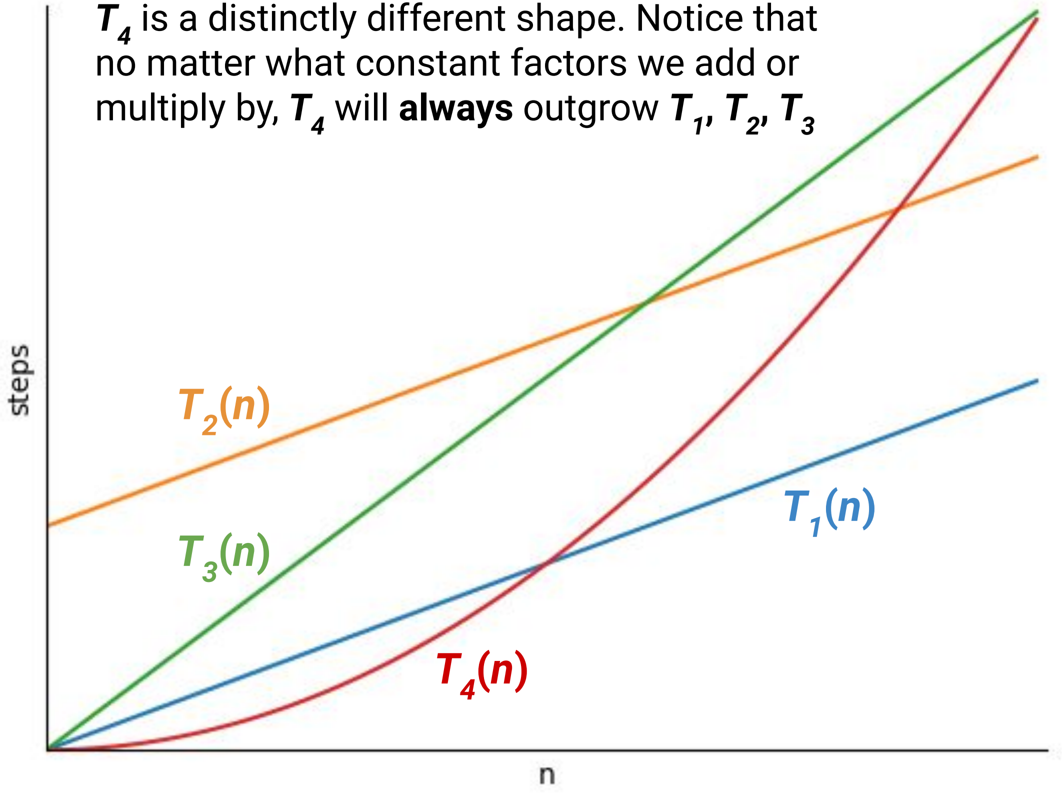


A Counter Example

Now consider:

$T_4(n) = n^2$

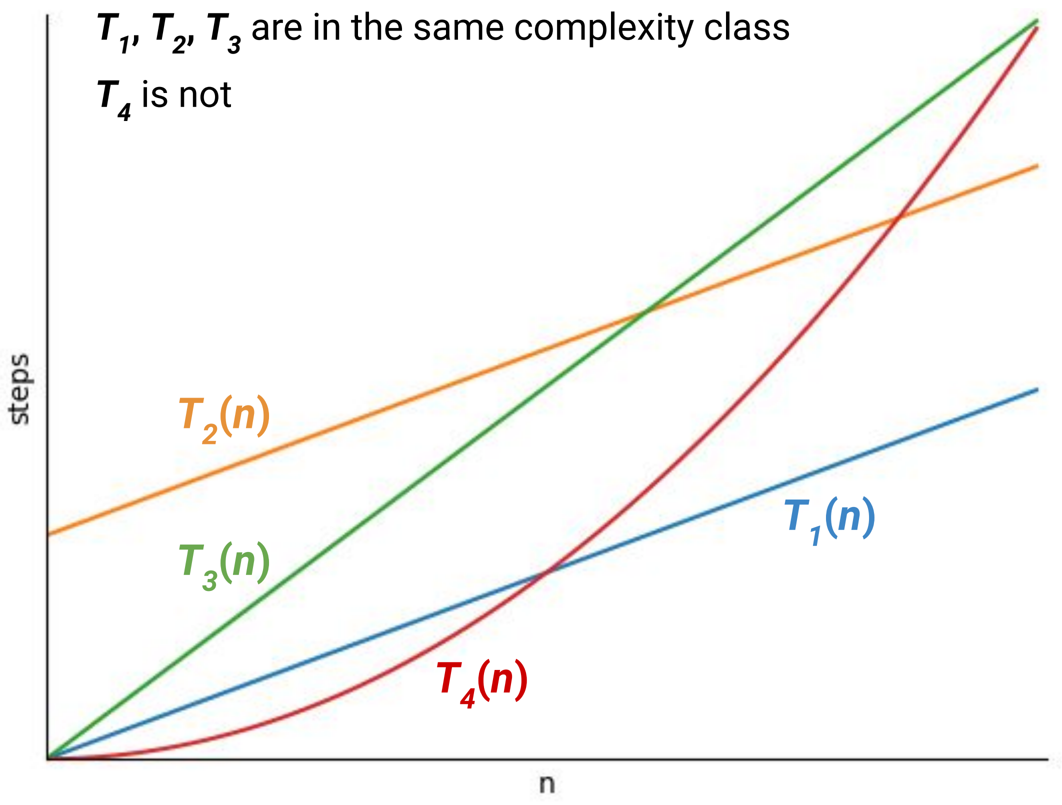
T_4 is a distinctly different shape. Notice that no matter what constant factors we add or multiply by, T_4 will **always** outgrow T_1, T_2, T_3



A Counter Example

Now consider:

$T_4(n) = n^2$



Complexity (so far…)

If there are constants c_1 and c_2 such that:

$$T_1(n) = c_1 + c_2T_2(n)$$

then we say T_1 and T_2 are in the same complexity class*

** not a complete definition…but we are getting there*

Back To Growth Functions

So what exactly counts as a step?

Back To Growth Functions

So what exactly counts as a step?

- An arithmetic operation
- Accessing a variable
- Printing to the screen
- etc

but…

Counting Steps

How many steps in each of these snippets?

```
x = 10;
```

```
x = 10;
y = 20;
```

Counting Steps

How many steps in each of these snippets?

```
x = 10;
```

$T_1(n) = 1$

```
x = 10;
y = 20;
```

Counting Steps

How many steps in each of these snippets?

```
x = 10;
```

$T_1(n) = 1$

```
x = 10;
y = 20;
```

$T_2(n) = 2$

Counting Steps

How many steps in each of these snippets?

```
x = 10;
```

$T_1(n) = 1$

```
x = 10;
y = 20;
```

$T_2(n) = 2$

$$T_2(n) = T_1(n) + 1$$

They are in the same complexity class…in 250 we treat them as the same

Counting Steps

A ***step*** therefore is ***any code*** that always has the same runtime

Notation - Big Theta

Θ(*f*(*n*)) is the **set** of all functions in the same complexity class as ***f***

Notation - Big Theta

$\Theta(f(n))$ is the **set** of all functions in the same complexity class as ***f***

Example: $\Theta(3n + 4) = \{$

n,

$n - 6$,

$15n$,

…

$\}$

Notation - Big Theta

$\Theta(f(n))$ is the **set** of all functions in the same complexity class as ***f***

Example: $\Theta(3n + 4) = \{$

n,

$n - 6$,

$15n$,

…

$\}$

$g(n) \in \Theta(f(n))$ means ***g*** and ***f*** are in the same complexity class

Common Shorthand

$g(n) = \Theta(f(n))$ is common shorthand for $g(n) \in \Theta(f(n))$

Common Shorthand

$g(n) = \Theta(f(n))$ is common shorthand for $g(n) \in \Theta(f(n))$

$g(n)$ is in $\Theta(f(n))$ is common shorthand for $g(n) \in \Theta(f(n))$

Common Shorthand

$g(n) = \Theta(f(n))$ is common shorthand for $g(n) \in \Theta(f(n))$

$g(n)$ is in $\Theta(f(n))$ is common shorthand for $g(n) \in \Theta(f(n))$

Algorithm Foo is in $\Theta(f(n))$ is common shorthand for $T(n) \in \Theta(f(n))$ where $T(n)$ is the growth function describing the runtime of Foo

Common Shorthand

$g(n) = \Theta(f(n))$ is common shorthand for $g(n) \in \Theta(f(n))$

$g(n)$ is in $\Theta(f(n))$ is common shorthand for $g(n) \in \Theta(f(n))$

Algorithm Foo is in $\Theta(f(n))$ is common shorthand for $T(n) \in \Theta(f(n))$ where $T(n)$ is the growth function describing the runtime of Foo

Moving forward: $f(n)$, $g(n)$, $f_1(n)$, etc will be used to name any mathematical function that's a growth function

$T(n)$, $T_1(n)$, etc will be used for growth functions for specific algorithms

Complexity Class Names

$\Theta(1)$: Constant

$\Theta(\log(n))$: Logarithmic

$\Theta(n)$: Linear

$\Theta(n \log(n))$: Log-Linear

$\Theta(n^2)$: Quadratic

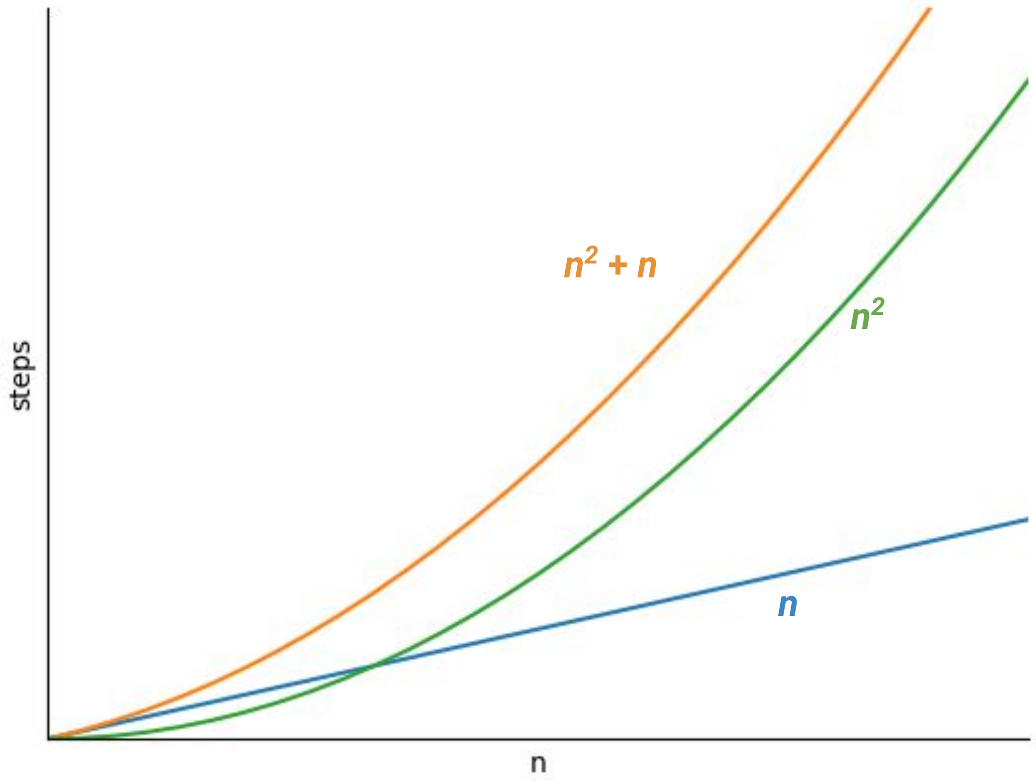
$\Theta(n^k)$: Polynomial

$\Theta(2^n)$: Exponential

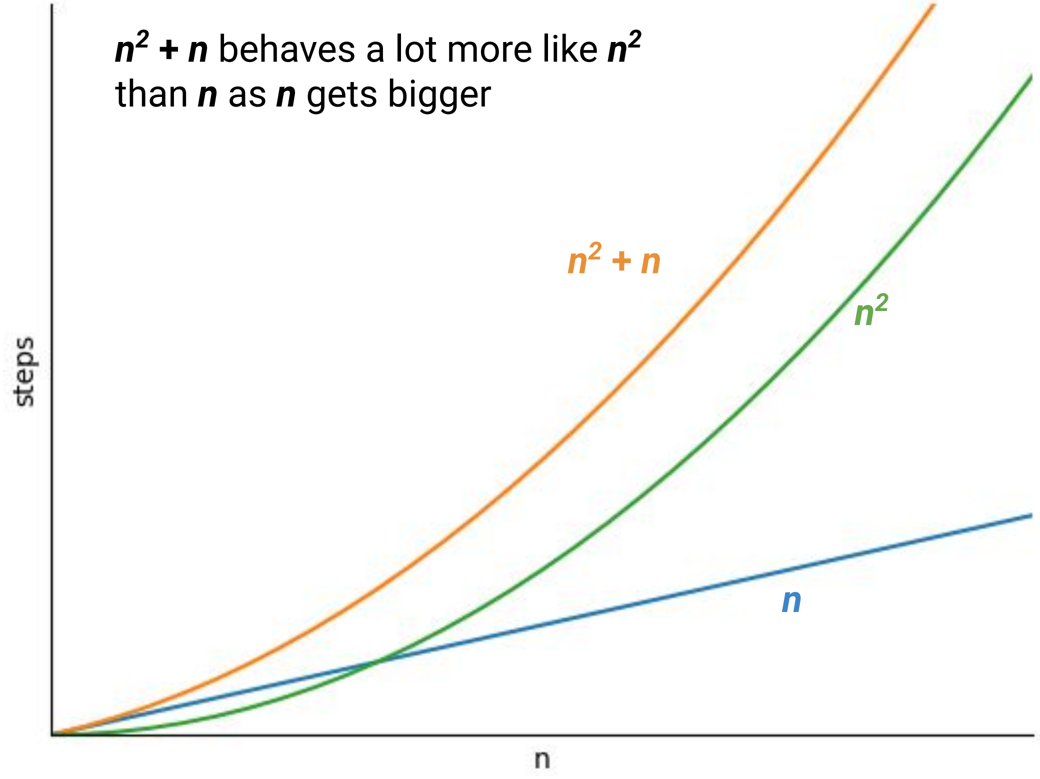
Combining Classes

What complexity class is $\boldsymbol{g(n) = n + n^2}$ in?

Combining Classes



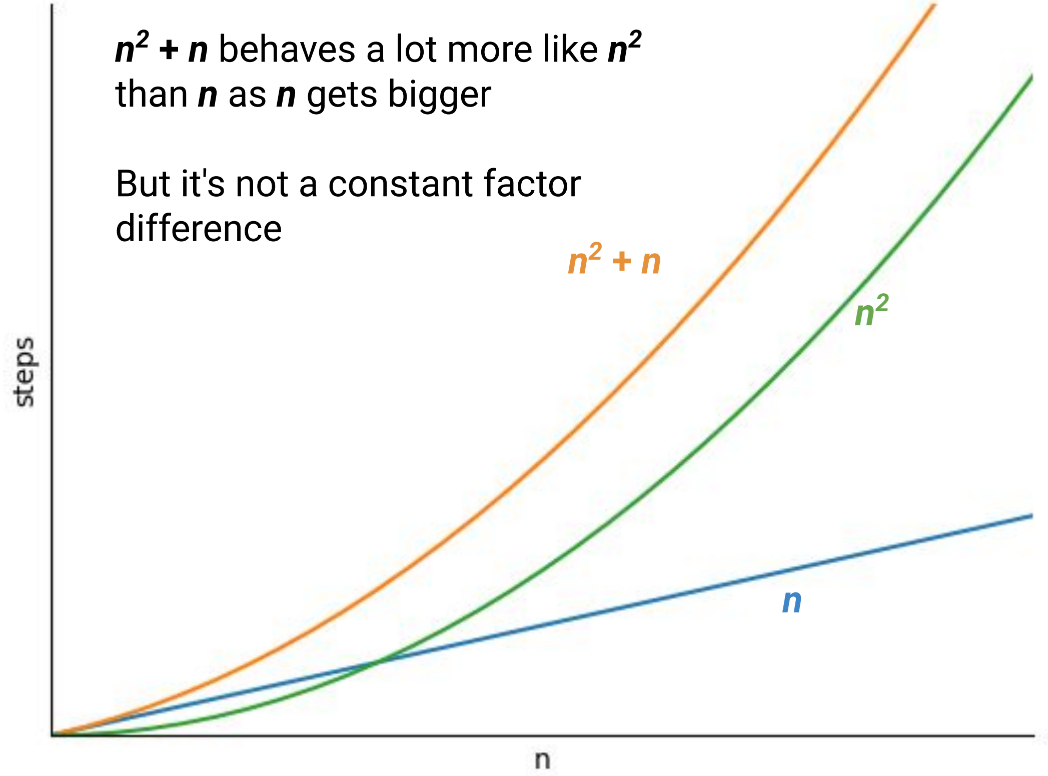
Combining Classes



Combining Classes

$n^2 + n$ behaves a lot more like n^2 than n as n gets bigger

But it's not a constant factor difference



Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

How does $n^2 + n$ relate to these two functions?

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$1 \leq n$$

remember, we only care about problems with non-negative input sizes

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$1 \leq n$$

$$n \leq n^2$$ multiply both sides by n

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$1 \leq n$$

$$n \leq n^2$$

$$n + n^2 \leq 2n^2$$

add n^2 to both sides

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$0 \leq n$$ obviously true

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$0 \leq n$$

$$n^2 \leq n + n^2$$ add n^2 to both sides

Combining Classes

Consider the fact that n^2 and $2n^2$ are in the same complexity class…

$$n^2 \le n + n^2 \le 2n^2$$

So $n^2 + n$ should probably be in $\Theta(n^2)$ too…

Complexity: A More Complete Definition

f and ***g*** are in the same complexity class iff:

g is bounded from above by something ***f***-shaped

and

g is bounded from below by something ***f***-shaped

Complexity: A More Complete Definition

f and ***g*** are in the same complexity class iff:

> ***g*** is bounded from above by something ***f***-shaped
>
> ***and***
>
> ***g*** is bounded from below by something ***f***-shaped

f shifted or stretched by a constant factor

Complexity: A More Complete Definition

f and ***g*** are in the same complexity class iff:

g is **bounded from above** by something ***f***-shaped

and

g is **bounded from below** by something ***f***-shaped

What do we mean by bounded from above/below?

Bounding from Above: Big O

***g*(*n*)** is bounded from above by ***f*(*n*)** if:

There exists a constant $n_0 > 0$ and a constant $c > 0$ such that:

For all $n > n_0$, $g(n) \leq c \cdot f(n)$

Bounding from Above: Big O

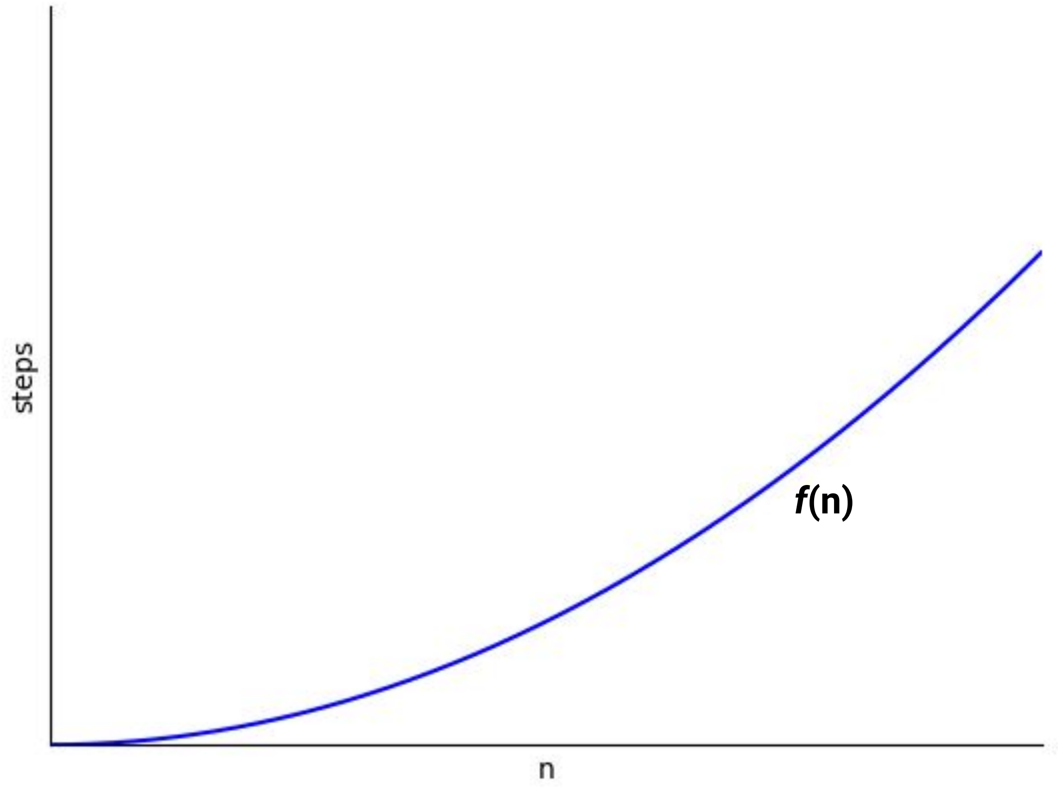
$g(n)$ is bounded from above by $f(n)$ if:

There exists a constant $n_0 > 0$ and a constant $c > 0$ such that:

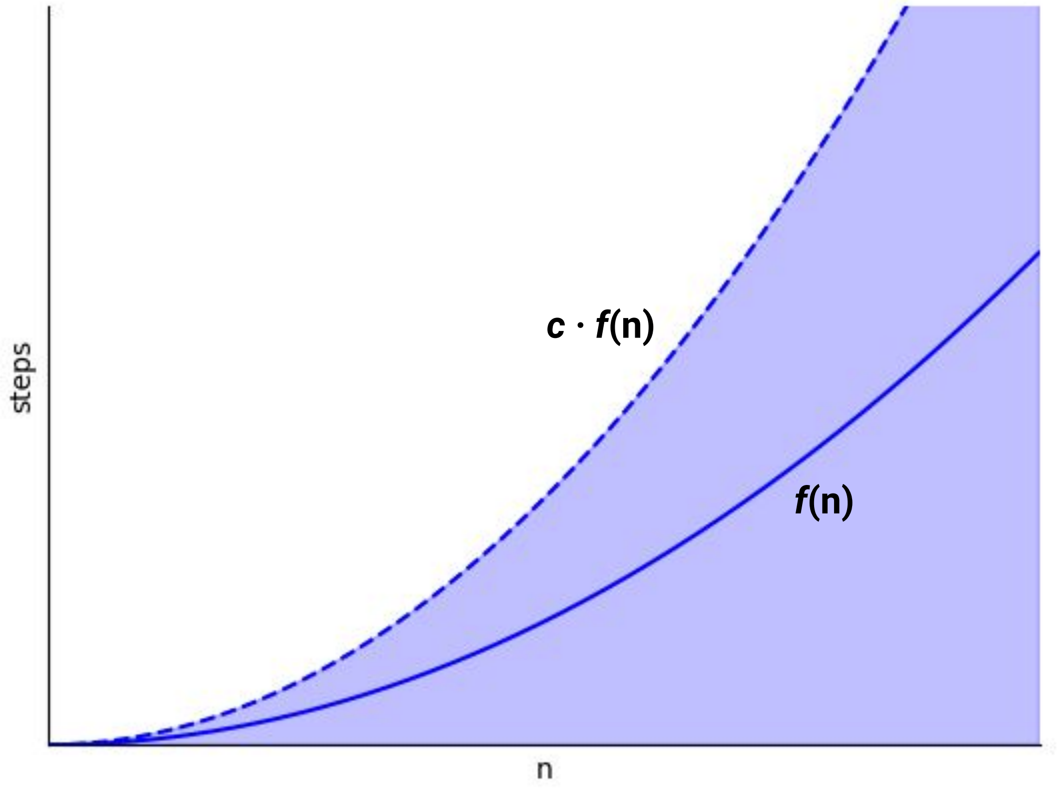
$$\text{For all } n > n_0,\ g(n) \leq c \cdot f(n)$$

In this case, we say that $g(n) \in O(f(n))$

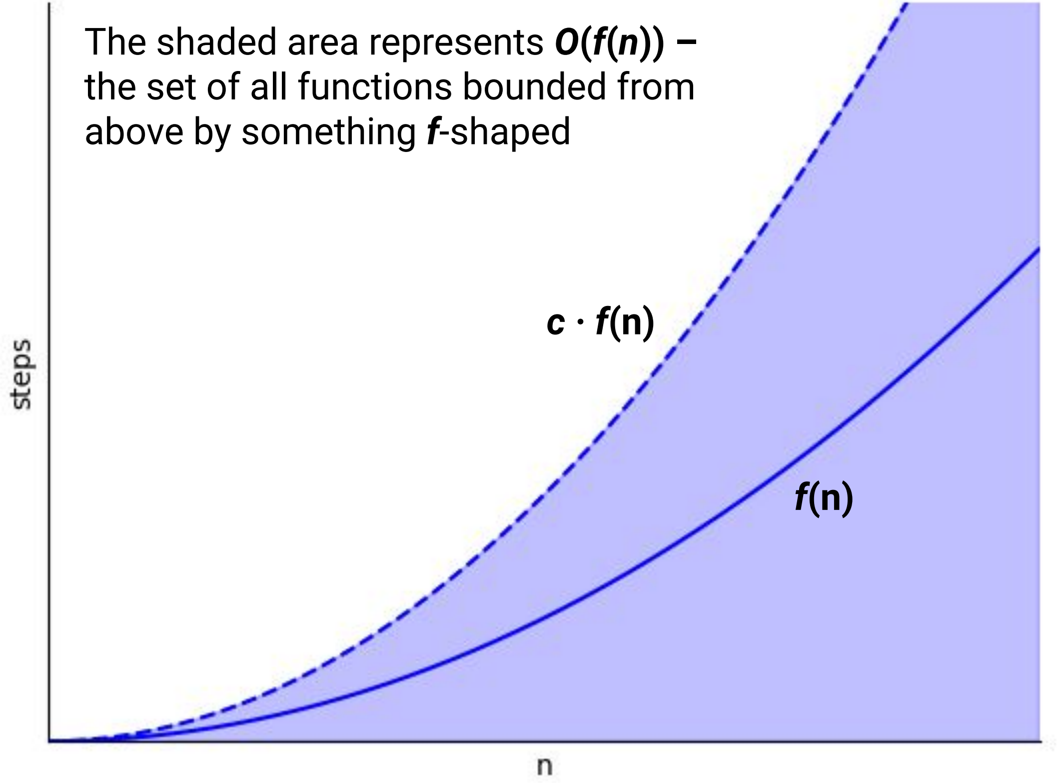
Bounded from Above: Big O



Bounded from Above: Big O



Bounded from Above: Big O



Bounding from Below: Big Omega

$g(n)$ is bounded from below by $f(n)$ if:

There exists a constant $n_0 > 0$ and a constant $c > 0$ such that:

$$\text{For all } n > n_0,\ g(n) \geq c \cdot f(n)$$

Bounding from Below: Big Omega

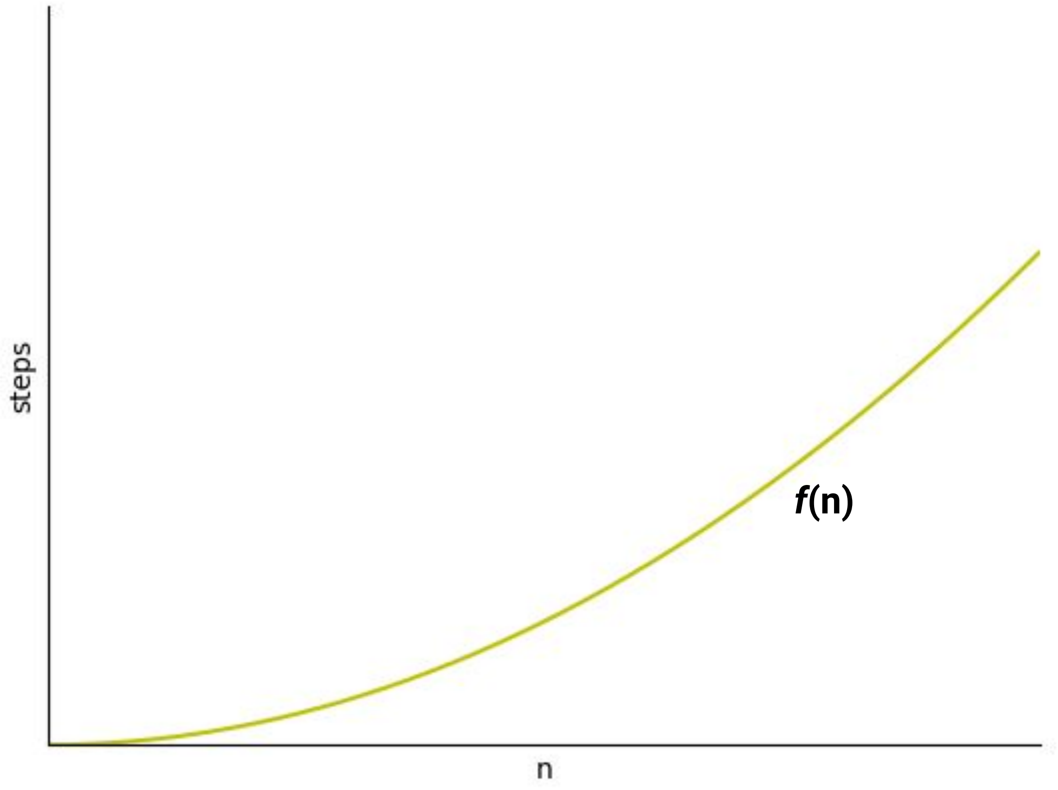
$g(n)$ is bounded from below by $f(n)$ if:

There exists a constant $n_0 > 0$ and a constant $c > 0$ such that:

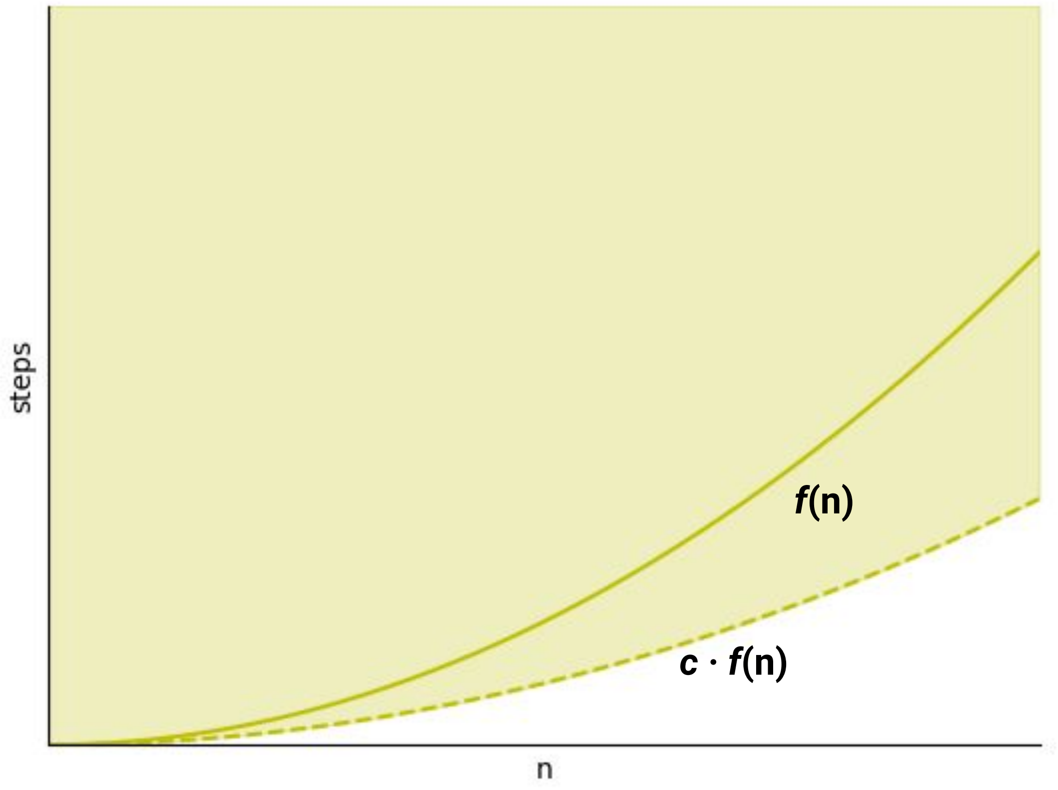
$$\text{For all } n > n_0,\ g(n) \geq c \cdot f(n)$$

In this case, we say that $g(n) \in \Omega(f(n))$

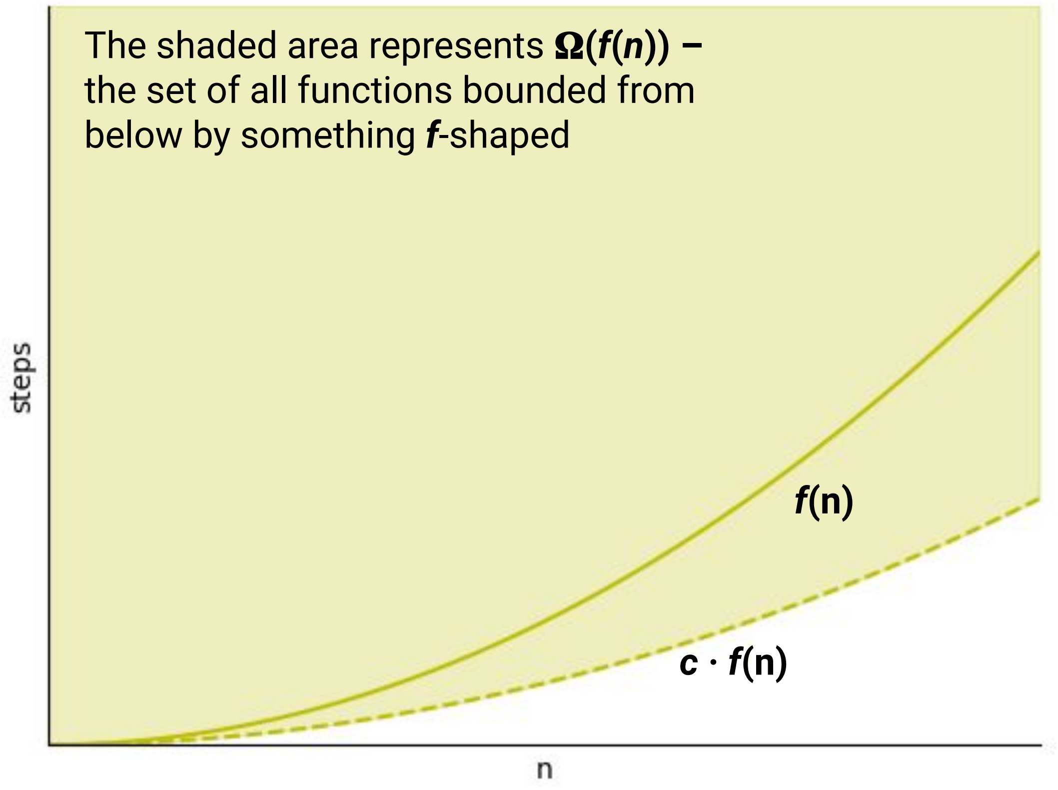
Bounded from Below: Big Ω



Bounded from Below: Big Ω



Bounded from Below: Big Ω



Complexity: A More Complete Definition

f and ***g*** are in the same complexity class iff:

g is bounded from above by something ***f***-shaped

and

g is bounded from below by something ***f***-shaped

Complexity: A More Complete Definition

***g*(*n*) ∈ Θ(*f*(*n*))** iff:

> ***g*** is bounded from above by something ***f***-shaped
>
> ***and***
>
> ***g*** is bounded from below by something ***f***-shaped

Complexity: A More Complete Definition

g(***n***) ∈ **Θ**(***f***(***n***)) iff:

g(***n***) ∈ ***O***(***f***(***n***))

and

g is bounded from below by something ***f***-shaped

Complexity: A More Complete Definition

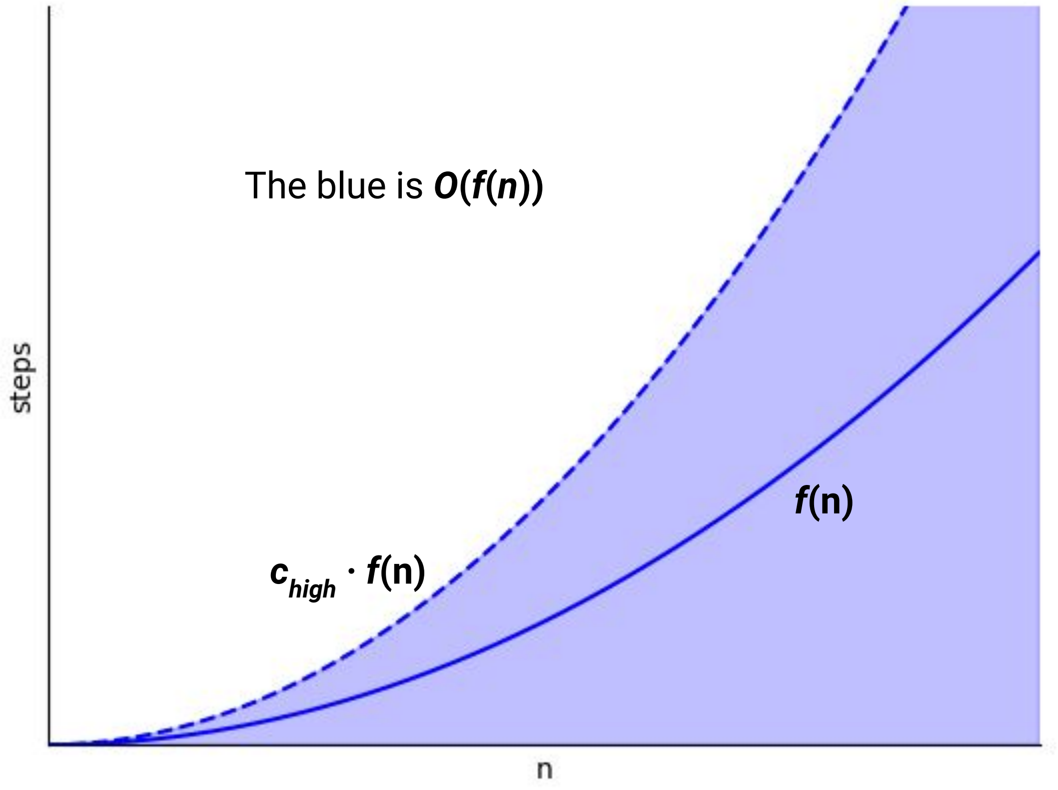
$g(n) \in \Theta(f(n))$ iff:

$g(n) \in O(f(n))$

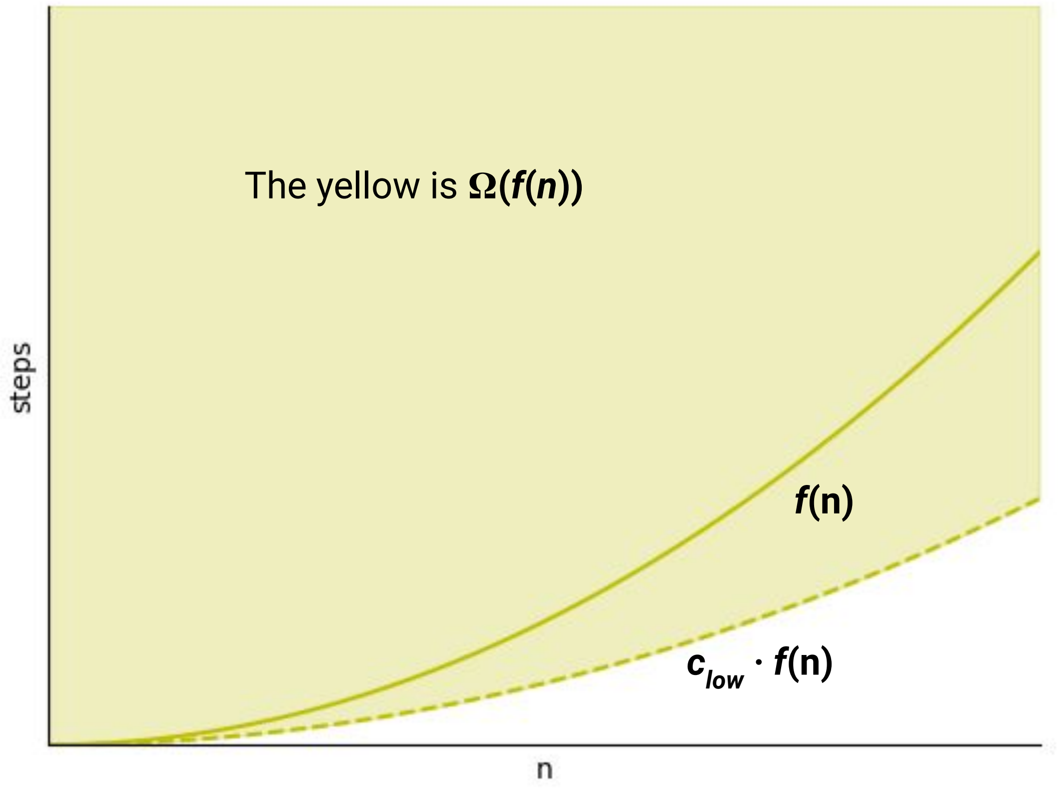
and

$g(n) \in \Omega(f(n))$

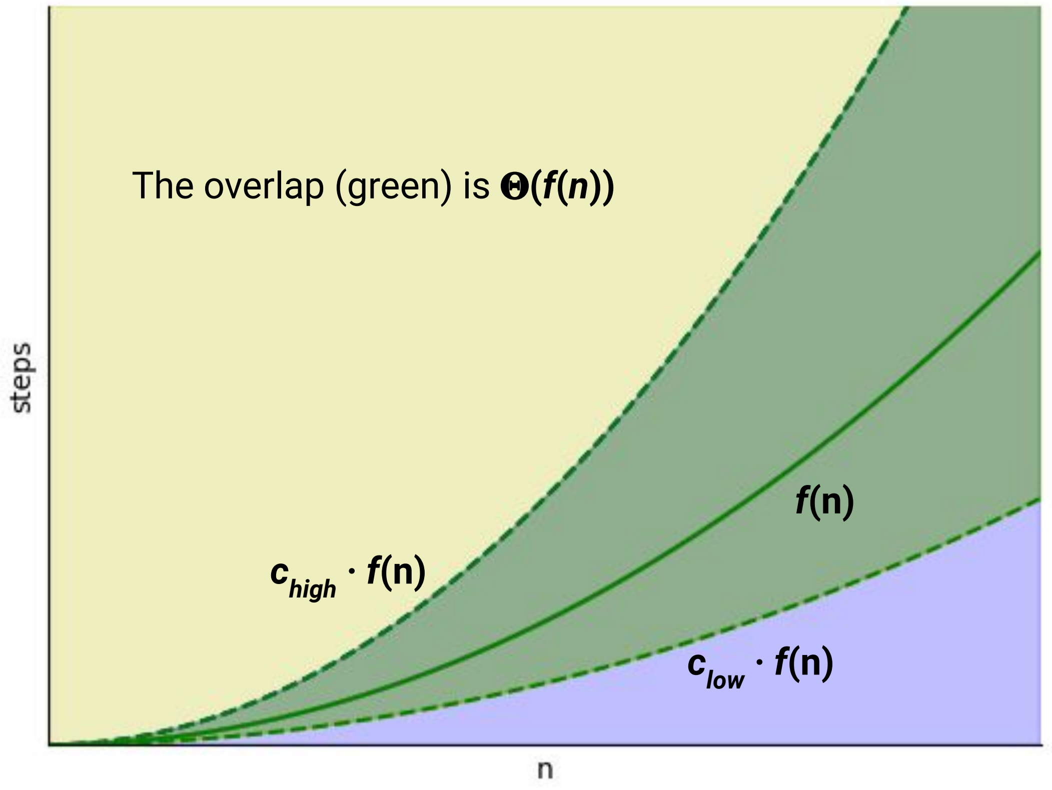
Complexity Class: Big Θ



Complexity Class: Big Θ

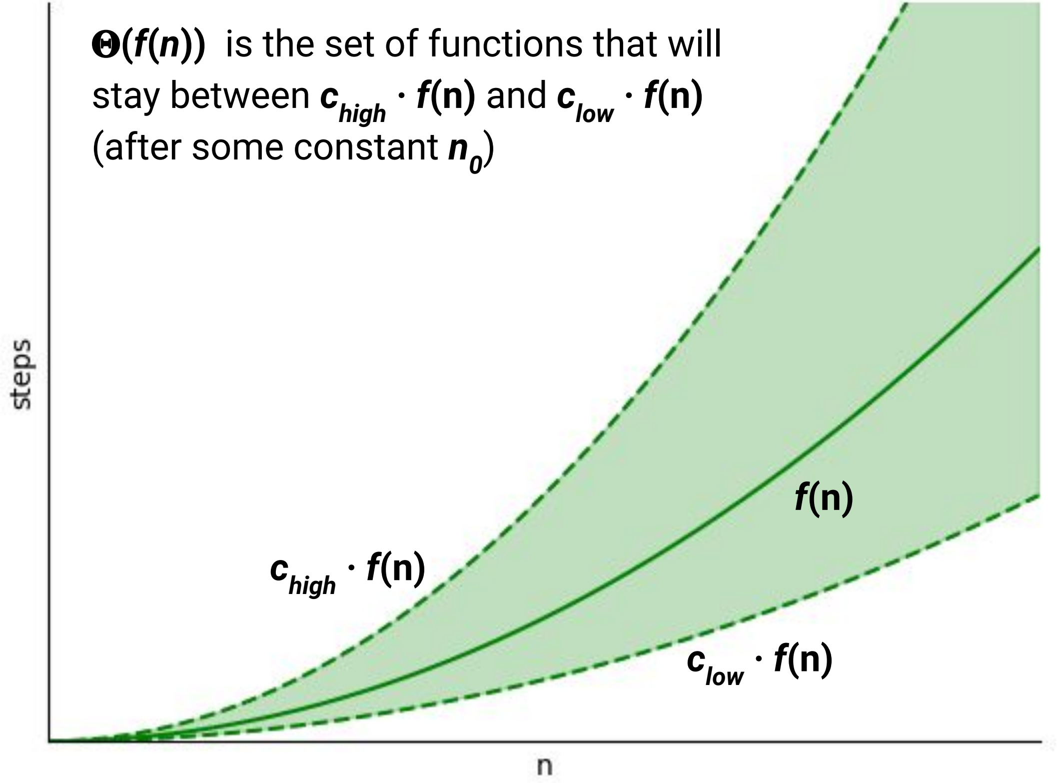


Complexity Class: Big Θ

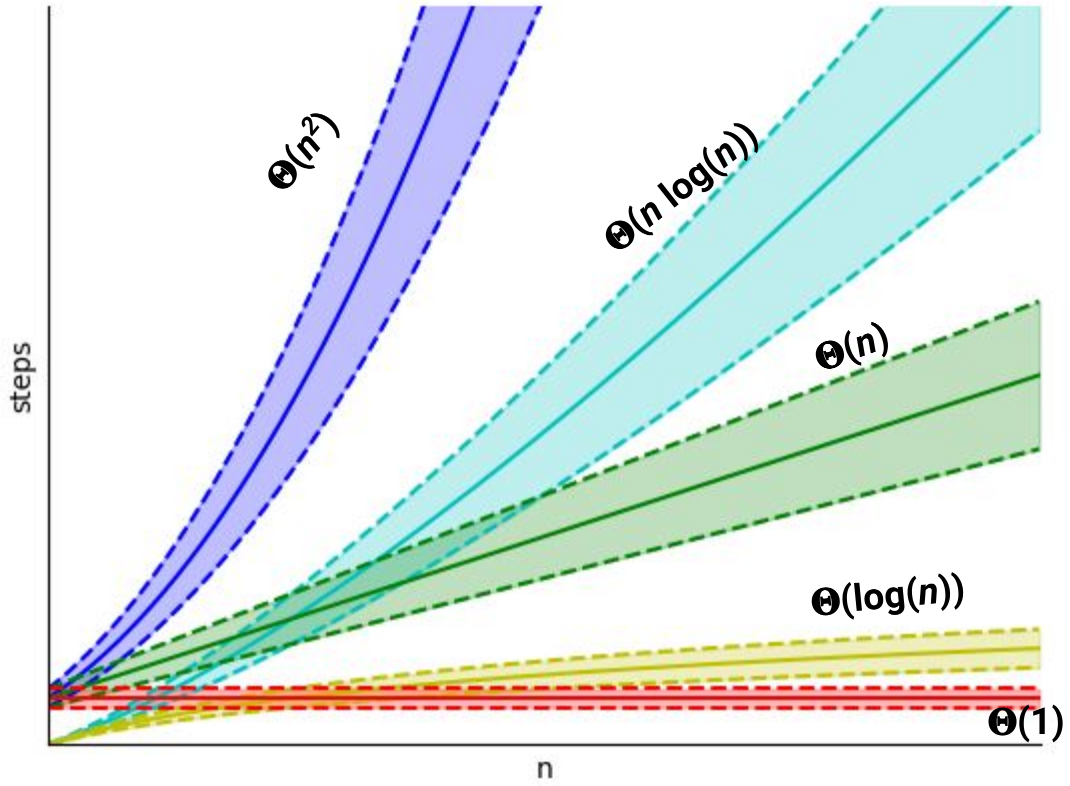


Complexity Class: Big Θ

Θ(***f***(***n***)) is the set of functions that will stay between $c_{high} \cdot f(n)$ and $c_{low} \cdot f(n)$ (after some constant n_0)



Complexity Class Ranking



$\Theta(1) < \Theta(\log(n)) < \Theta(n) < \Theta(n \log(n)) < \Theta(n^2) < \Theta(n^3) < \Theta(2^n)$

Rules of Thumb

$$\Theta(1) < \Theta(\log(n)) < \Theta(n) < \Theta(n \log(n)) < \Theta(n^2) < \Theta(n^3) < \Theta(2^n)$$

Rules of Thumb

$$O(1) \subset O(\log(n)) \subset O(n) \subset O(n \log(n)) \subset O(n^2) \subset O(n^3) \subset O(2^n)$$

$$\Omega(2^n) \subset \Omega(n^3) \subset \Omega(n^2) \subset \Omega(n \log(n)) \subset \Omega(n) \subset \Omega(\log(n)) \subset \Omega(1)$$

Rules of Thumb

If something is bounded from above by **log(*n*)**, it's also bounded from above by ***n***

$$O(1) \subset O(\log(n)) \subset O(n) \subset O(n \log(n)) \subset O(n^2) \subset O(n^3) \subset O(2^n)$$

$$\Omega(2^n) \subset \Omega(n^3) \subset \Omega(n^2) \subset \Omega(n \log(n)) \subset \Omega(n) \subset \Omega(\log(n)) \subset \Omega(1)$$

Rules of Thumb

$$O(1) \subset O(\log(n)) \subset O(n) \subset O(n \log(n)) \subset O(n^2) \subset O(n^3) \subset O(2^n)$$

$$\Omega(2^n) \subset \Omega(n^3) \subset \Omega(n^2) \subset \Omega(n \log(n)) \subset \Omega(n) \subset \Omega(\log(n)) \subset \Omega(1)$$

If something is bounded from below by n^2, it's also bounded from below by n

Rules of Thumb

$O(f(n))$ (Big-O): The complexity class of $f(n)$ and every lesser class

$\Theta(f(n))$ (Big-Θ): The complexity class of $f(n)$

$\Omega(f(n))$ (Big-Ω): The complexity class of $f(n)$ and every greater class

$$O$$

$$\theta$$

$$\Omega$$