CSE 250
Data Structures

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Lec 06: Proving Bounds

Announcements and Feedback

- PA1 released
  - Testing phase due Sunday 2/11 @ 11:59PM
    - Recitation this week will go over tips for testing
    - AutoLab should be up later tonight
    - Try building/running your tests before submitting!
  - Implementation phase due Sunday 2/18 @ 11:59PM

"Shape" of a Function

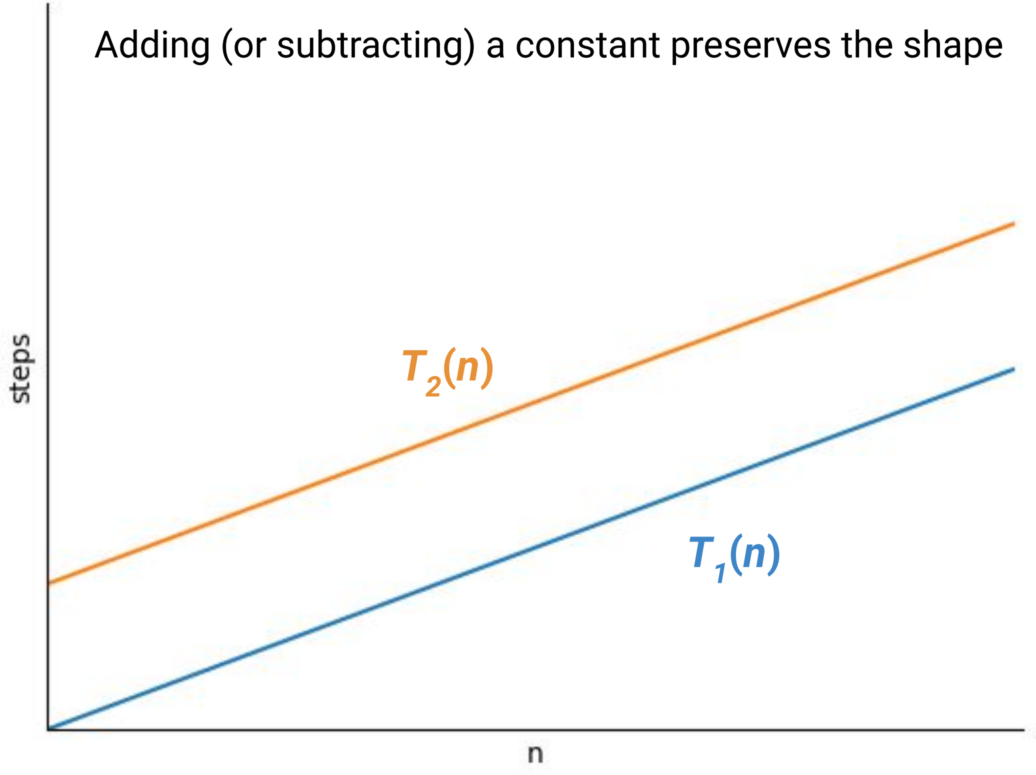
When do we consider two functions to have the same shape?

Additive Factors

Consider two growth functions:

$T_1(n) = 3n$

$T_2(n) = 3n + 3$

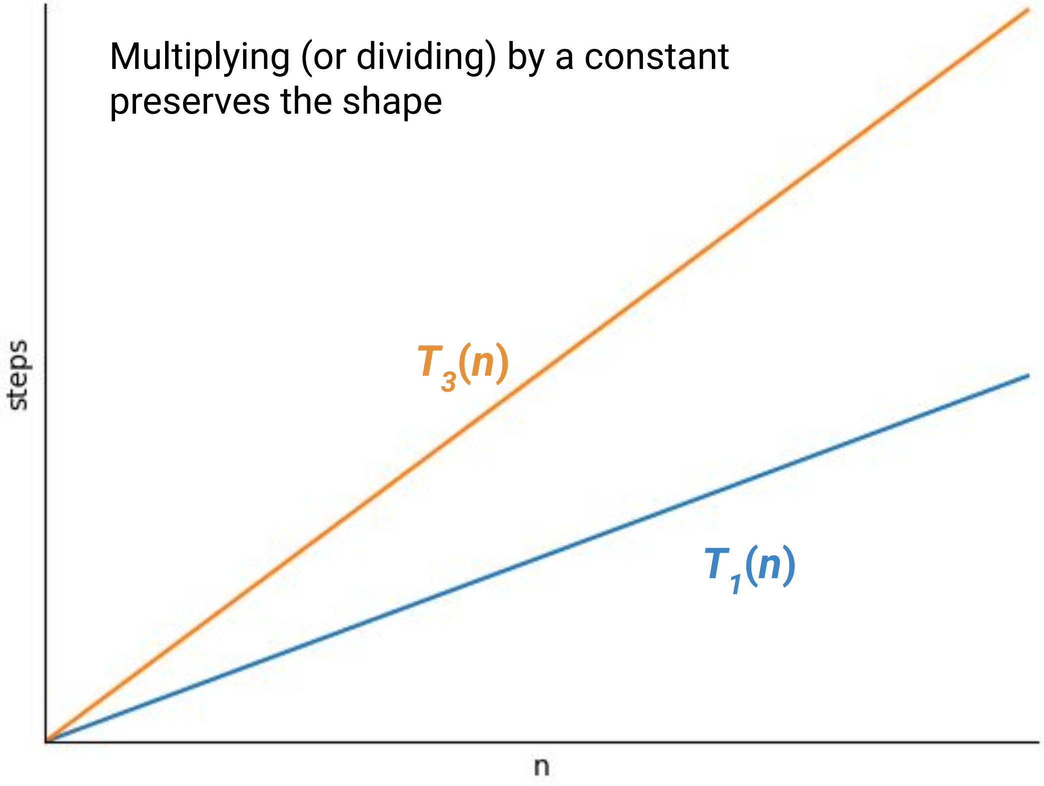


Multiplicative Factors

Consider two growth functions:

$T_1(n) = 3n$

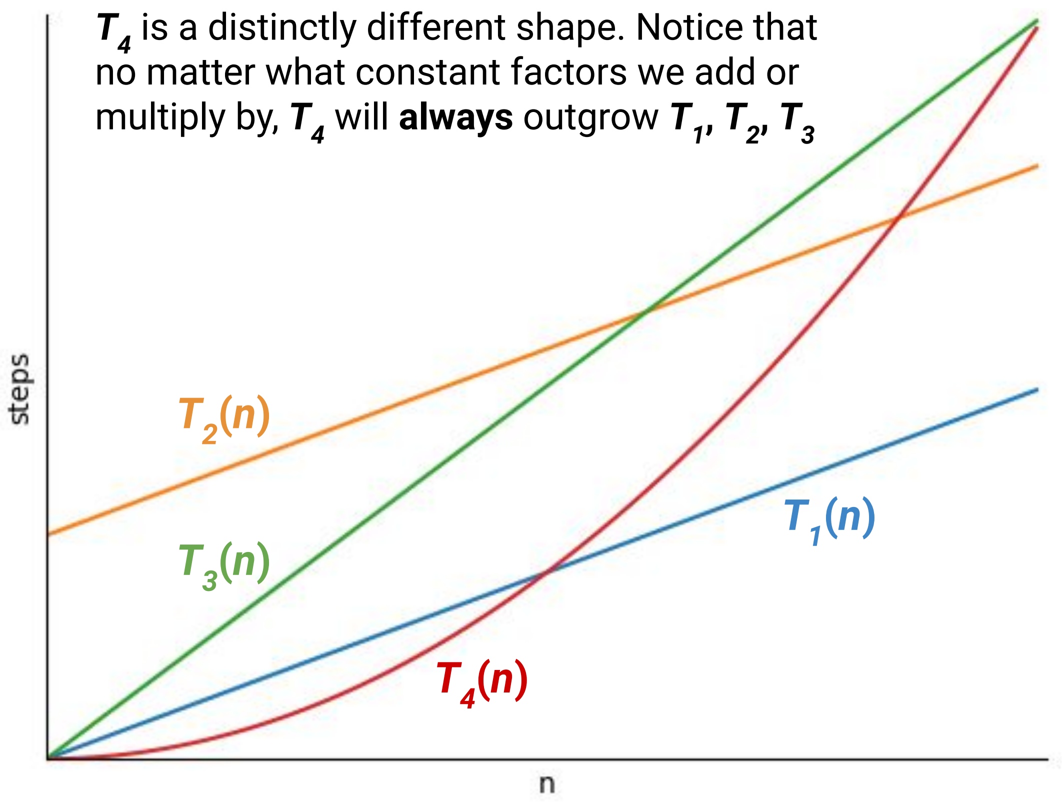
$T_3(n) = 6n$



A Counter Example

Now consider:

$T_4(n) = n^2$



Complexity Class

f and ***g*** are in the same complexity class, denoted $\boldsymbol{g(n) \in \Theta(f(n))}$, iff:

g is bounded from above by something ***f***-shaped

and

g is bounded from below by something ***f***-shaped

Bounded from Above: Big O

$g(n)$ is bounded from above by $f(n)$ if:

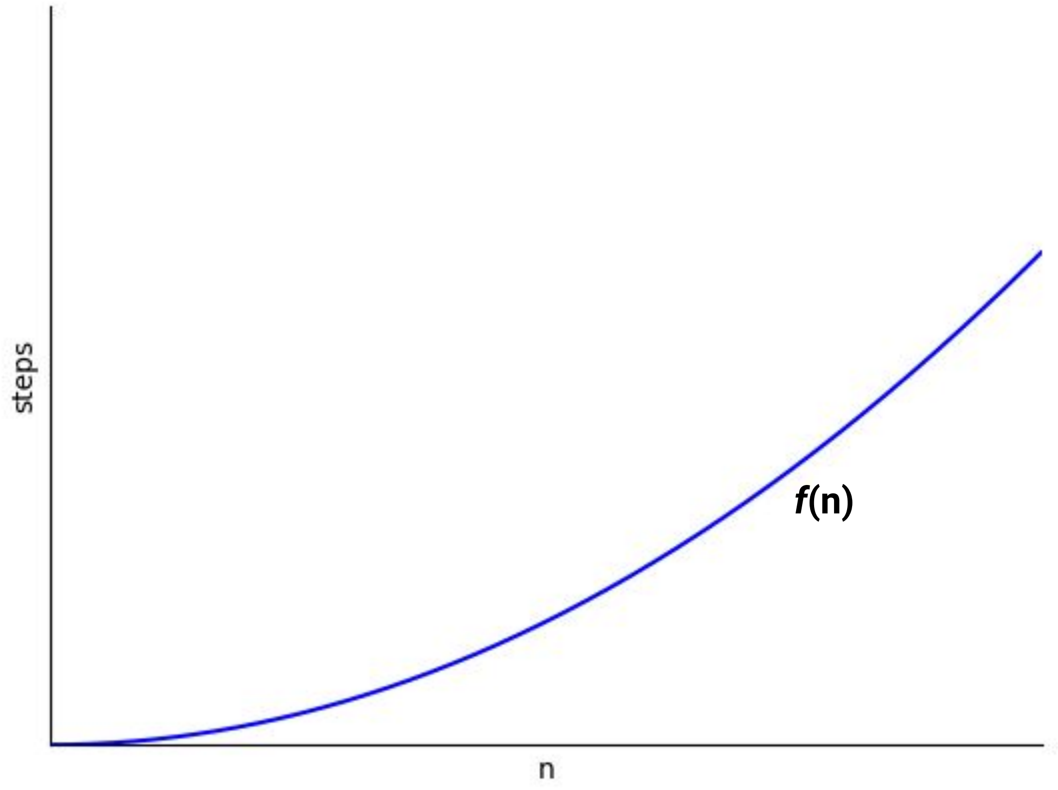
There exists a constant $n_0 > 0$ and a constant $c > 0$ such that:

$$\text{For all } n > n_0,\ g(n) \le c \cdot f(n)$$

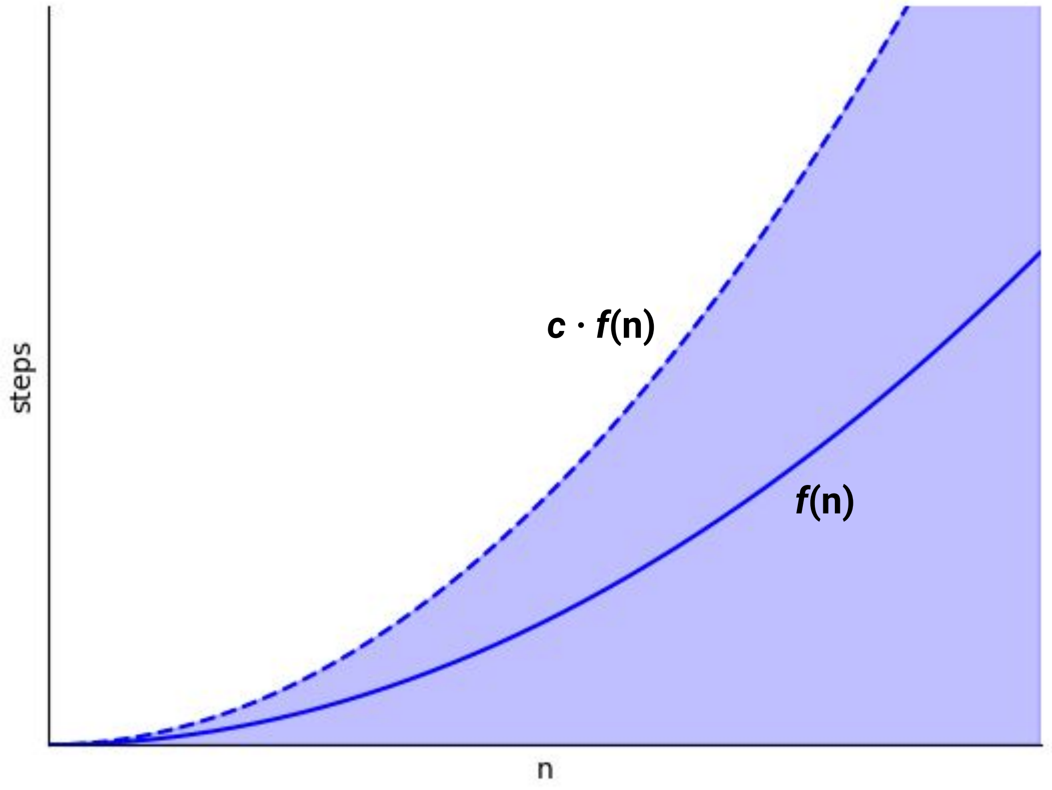
In this case, we say that $g(n) \in O(f(n))$

$O(f(n))$ is the set of all functions bounded from above by $f(n)$

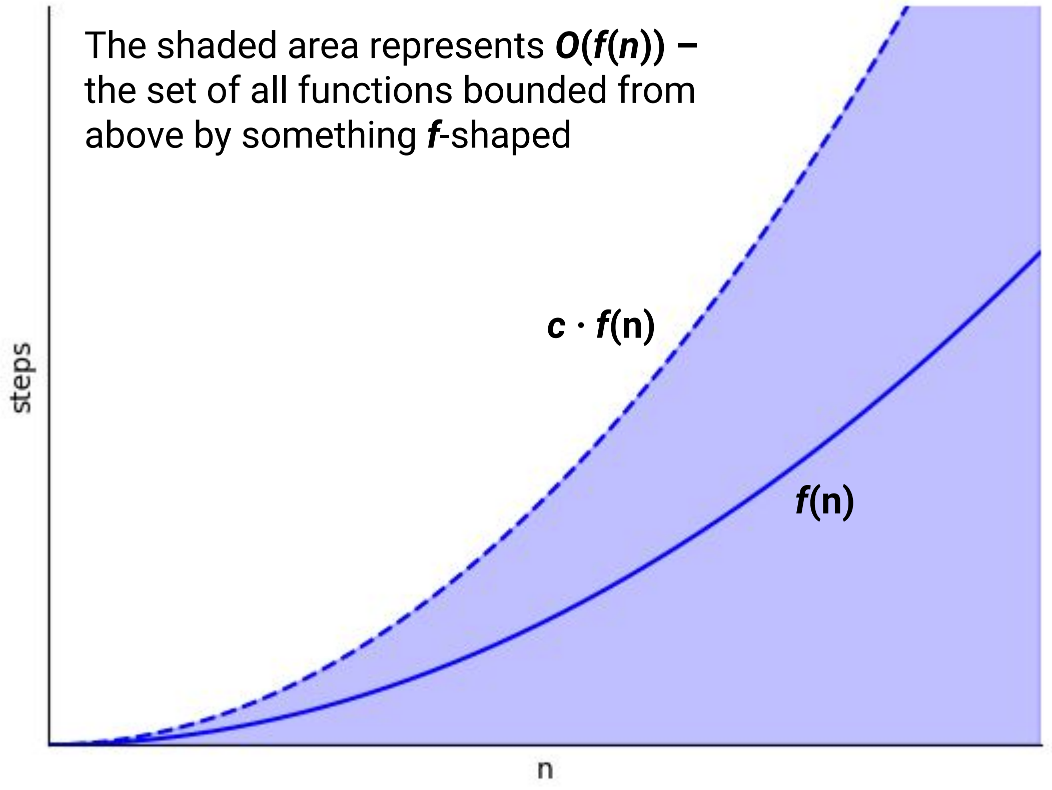
Bounded from Above: Big O



Bounded from Above: Big O



Bounded from Above: Big O



Bounded from Above: Big O

$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that $\boldsymbol{g(n) \in O(n^2)}$

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

Inequality Tricks

1. $f(n) \geq g(n)$ is true if $f(n)/a \geq g(n)/a$ (for any $a > 0$)
2. $f(n) \geq g(n)$ is true if $f(n)*a \geq g(n)*a$ (for any $a > 0$)
3. $x + a \geq y + b$ is true if $x \geq y$ and $a \geq b$ (for any a, b)
4. $x \geq y$ is true if $x \geq a$ and $a \geq y$ (for any a)
5. $1 \leq \log(n) \leq n \leq n^2 \leq n^k$ (for $k \geq 2$) $\leq 2^n$

Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq c_1 \cdot n^2 \qquad 4n \leq c_2 \cdot n^2 \qquad 7 \leq c_3 \cdot n^2$$

Bounded from Above: Big O

$$\frac{n^2}{2} \leq c_1 \cdot n^2$$

Bounded from Above: Big O

$$\frac{n^2}{2} \leq c_1 \cdot n^2$$

This is true for all $\boldsymbol{n} \geq \mathbf{0}$ if we set c_1 to **1/2**

Bounded from Above: Big O

$$4n \leq c_2 \cdot n^2$$

Bounded from Above: Big O

$$4n \leq c_2 \cdot n^2$$

This is true for all $\boldsymbol{n} \geq \mathbf{0}$ if we set $\boldsymbol{c_2}$ to **4**

Bounded from Above: Big O

$$7 \leq c_3 \cdot n^2$$

Bounded from Above: Big O

$$7 \leq c_3 \cdot n^2$$

This is true for all $\boldsymbol{n} \geq \mathbf{1}$ if we set $\boldsymbol{c_3}$ to **7**

Bounded from Above: Big O

$$\boxed{\frac{n^2}{2}} + \boxed{4n} + \boxed{7} \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq c_1 \cdot n^2 \qquad 4n \leq c_2 \cdot n^2 \qquad 7 \leq c_3 \cdot n^2$$

Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq \frac{1}{2} \cdot n^2 \qquad 4n \leq 4 \cdot n^2 \qquad 7 \leq 7 \cdot n^2$$

Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq \frac{1}{2} \cdot n^2 \qquad 4n \leq 4 \cdot n^2 \qquad 7 \leq 7 \cdot n^2$$

$$\frac{n^2}{2} + 4n + 7 \leq \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2$$

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First show:

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$$\frac{n^2}{2} + 4n + 7 \leq \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2$$

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$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq \frac{1}{2} \cdot n^2 \qquad 4n \leq 4 \cdot n^2 \qquad 7 \leq 7 \cdot n^2$$

$$\frac{n^2}{2} + 4n + 7 \leq \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2$$

Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

First show:

$$\frac{n^2}{2} \leq \frac{1}{2} \cdot n^2 \qquad 4n \leq 4 \cdot n^2 \qquad 7 \leq 7 \cdot n^2$$

$$\frac{n^2}{2} + 4n + 7 \leq \frac{1}{2} \cdot n^2 + 4 \cdot n^2 + 7 \cdot n^2 = (\frac{1}{2} + 4 + 7) \cdot n^2$$

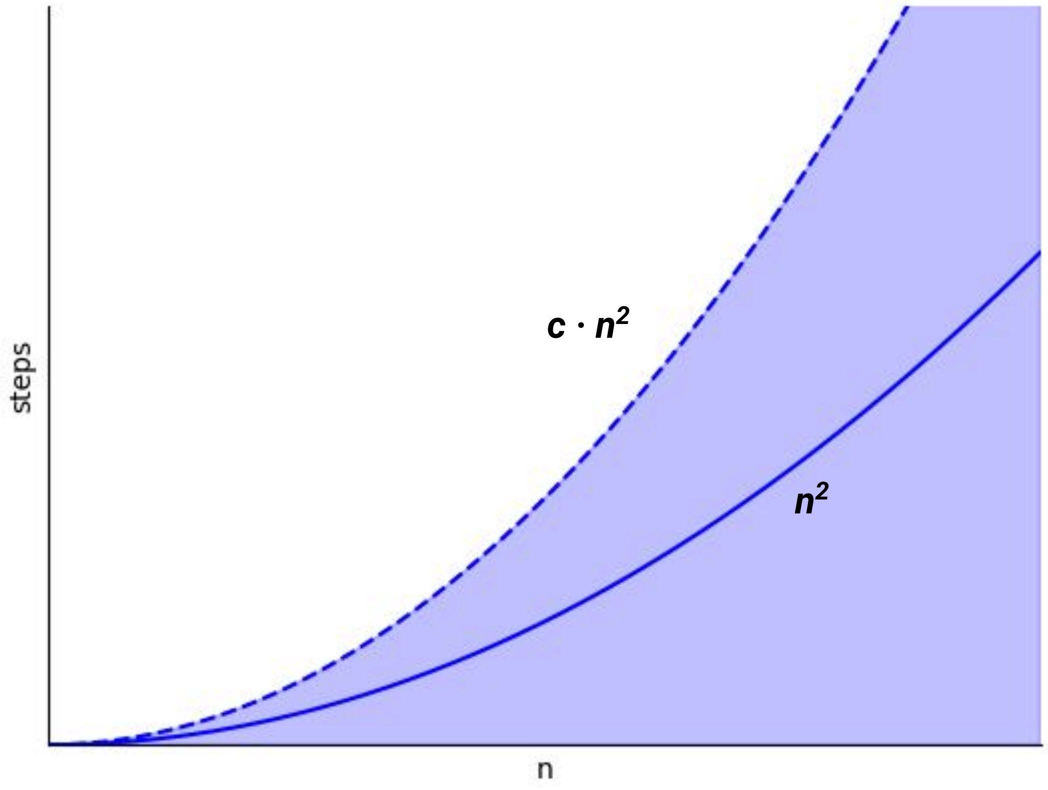
Bounded from Above: Big O

$$\frac{n^2}{2} + 4n + 7 \leq c \cdot n^2$$

Therefore if we let ***c* = 11.5**, then for all ***n* ≥ 1** the above holds true

Therfore ***g*(*n*) ∈ *O*(*n*²)**

Bounded from Above: Big O



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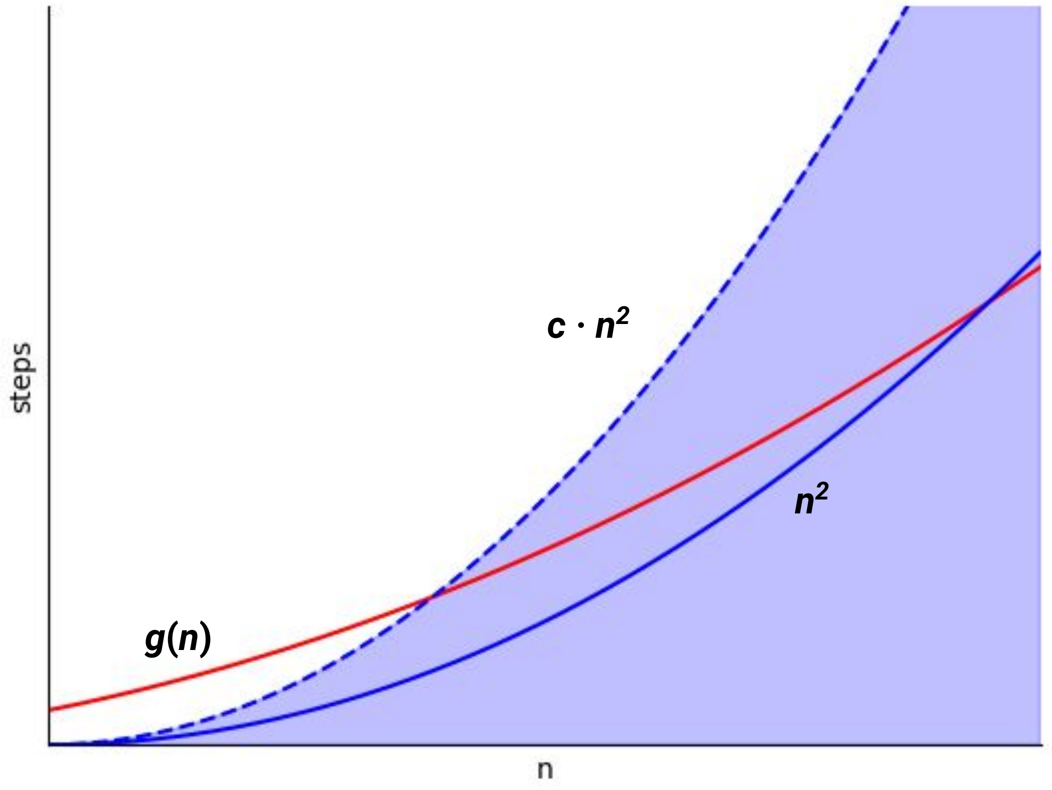

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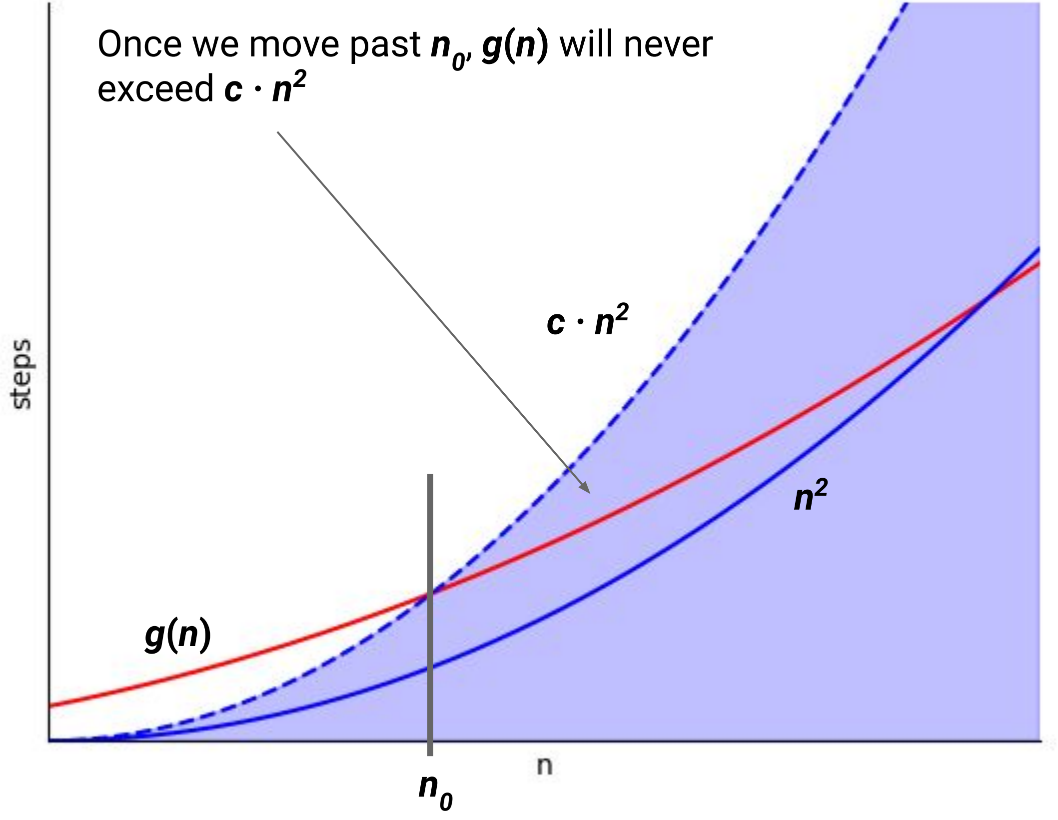

Bounded from Below: Big Ω

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There exists a constant $n_0 > 0$ and a constant $c > 0$ such that:

$$\text{For all } n > n_0,\ g(n) \geq c \cdot f(n)$$

In this case, we say that $g(n) \in \Omega(f(n))$

$\Omega(f(n))$ is the set of all functions bounded from below by $f(n)$

Bounded from Below: Big Ω

$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that $\boldsymbol{g(n) \in \Omega(n^2)}$

$$\frac{n^2}{2} + 4n + 7 \geq c \cdot n^2$$

Bounded from Below: Big Ω

$$\boxed{\frac{n^2}{2}} + 4n + 7 \geq c \cdot n^2$$

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

We'll start with a similar approach...

Bounded from Below: Big Ω

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

Bounded from Below: Big Ω

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

This is true for all $n \geq 0$ if we set c_1 to **1/2**

Bounded from Below: Big Ω

$$\frac{n^2}{2} + 4n + 7 \geq c \cdot n^2$$

Now that we've shown this…what else do we need to show for the overall equation to be true?

$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

Bounded from Below: Big Ω

$$\frac{n^2}{2} + 4n + 7 \geq c \cdot n^2$$

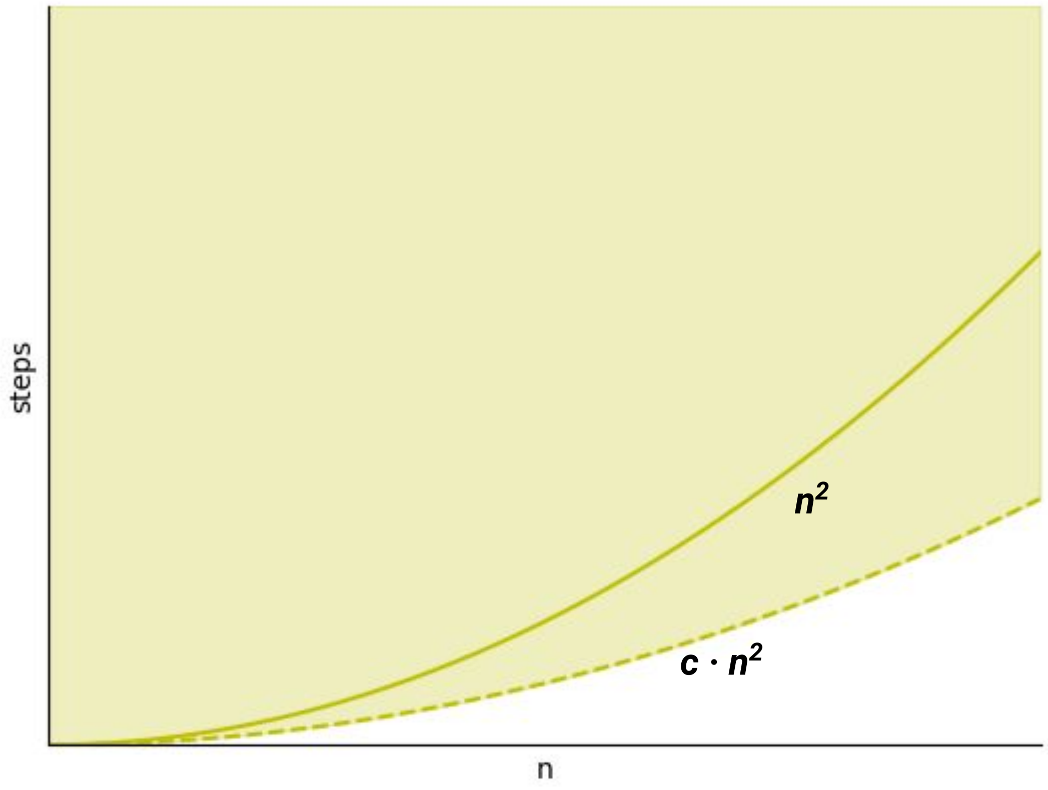
$$\frac{n^2}{2} \geq c_1 \cdot n^2$$

Now that we've shown this…what else do we need to show for the overall equation to be true?

Just need to show that 4n and 7 are ≥ 0

By adding non-negative things to the first term we can only make it bigger!

Bounded from Below: Big Ω



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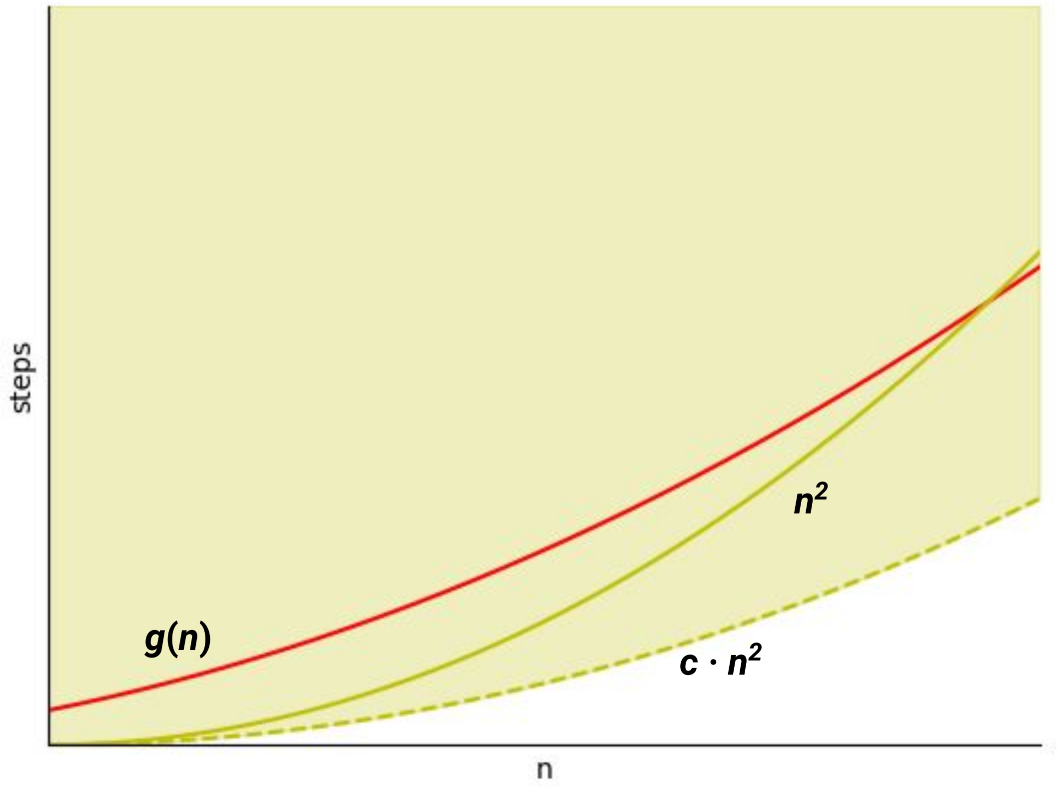

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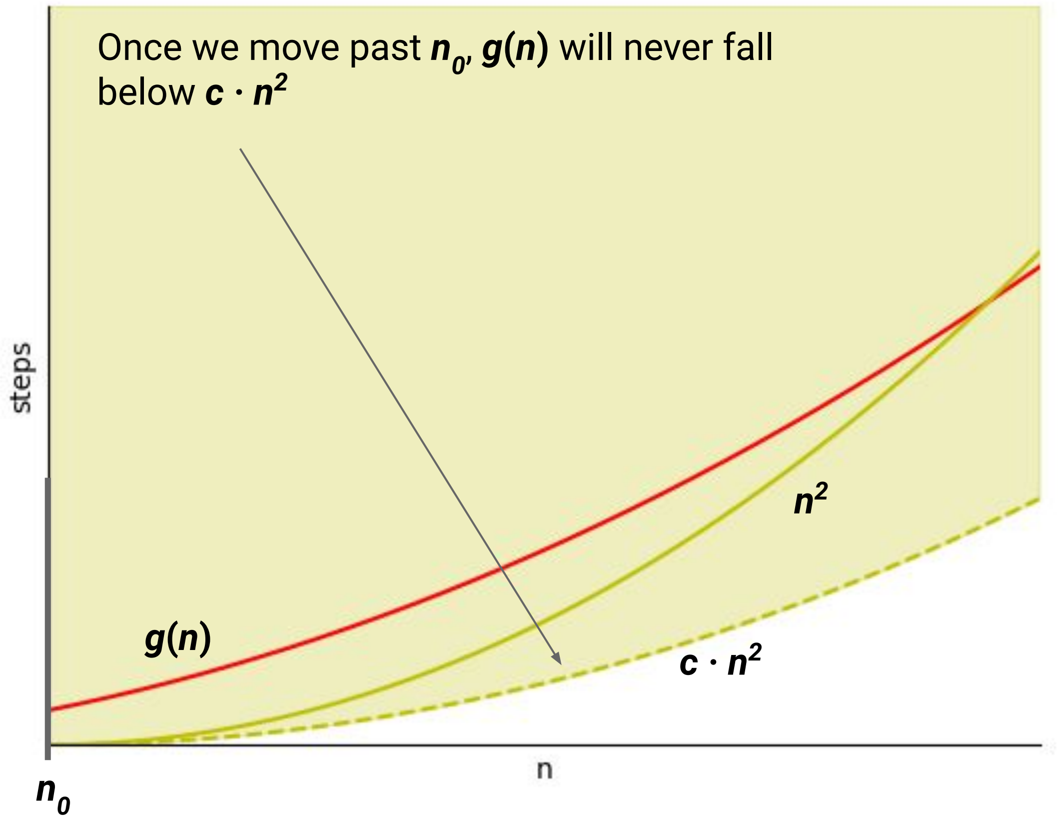

Complexity Class: Big Θ

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and

$g(n) \in \Omega(f(n))$

Bounded from Below: Big Θ

$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that $\boldsymbol{g(n) \in \Theta(n^2)}$

Bounded from Below: Big Θ

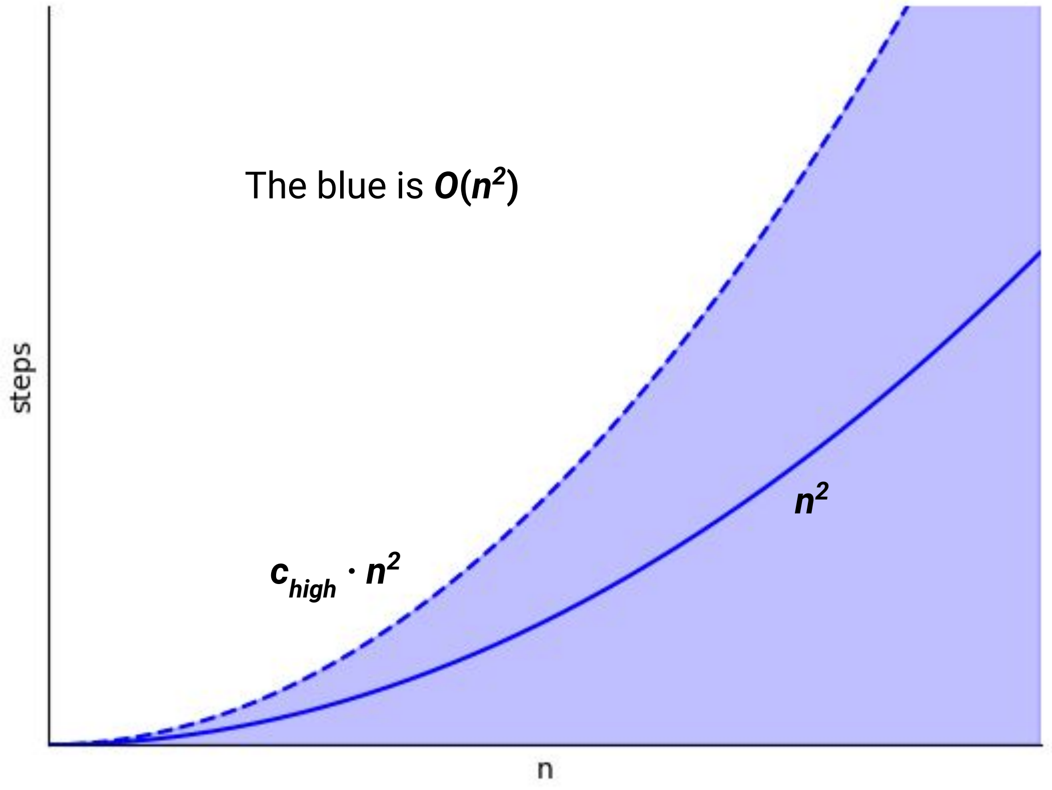
$$g(n) = \frac{n^2}{2} + 4n + 7$$

Prove that $g(n) \in \Theta(n^2)$

We just proved that $g(n) \in O(n^2)$ and $g(n) \in \Omega(n^2)$

Therefore we have proved that $g(n) \in \Theta(n^2)$

Complexity Class: Big Θ



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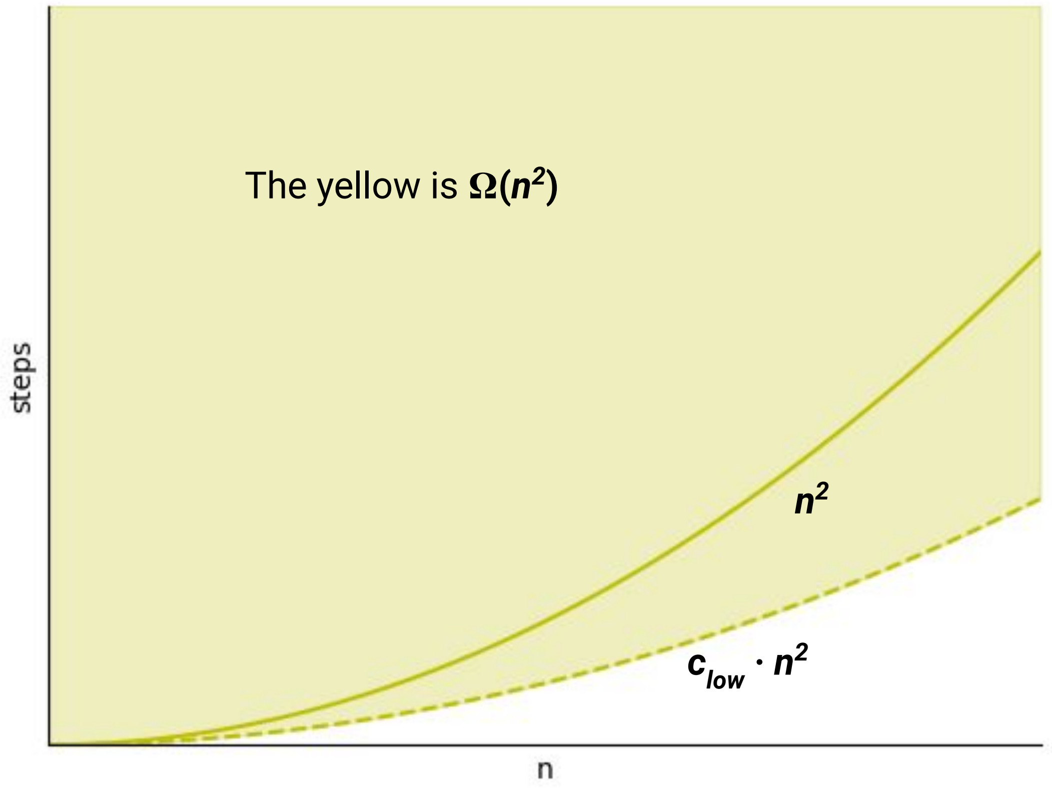

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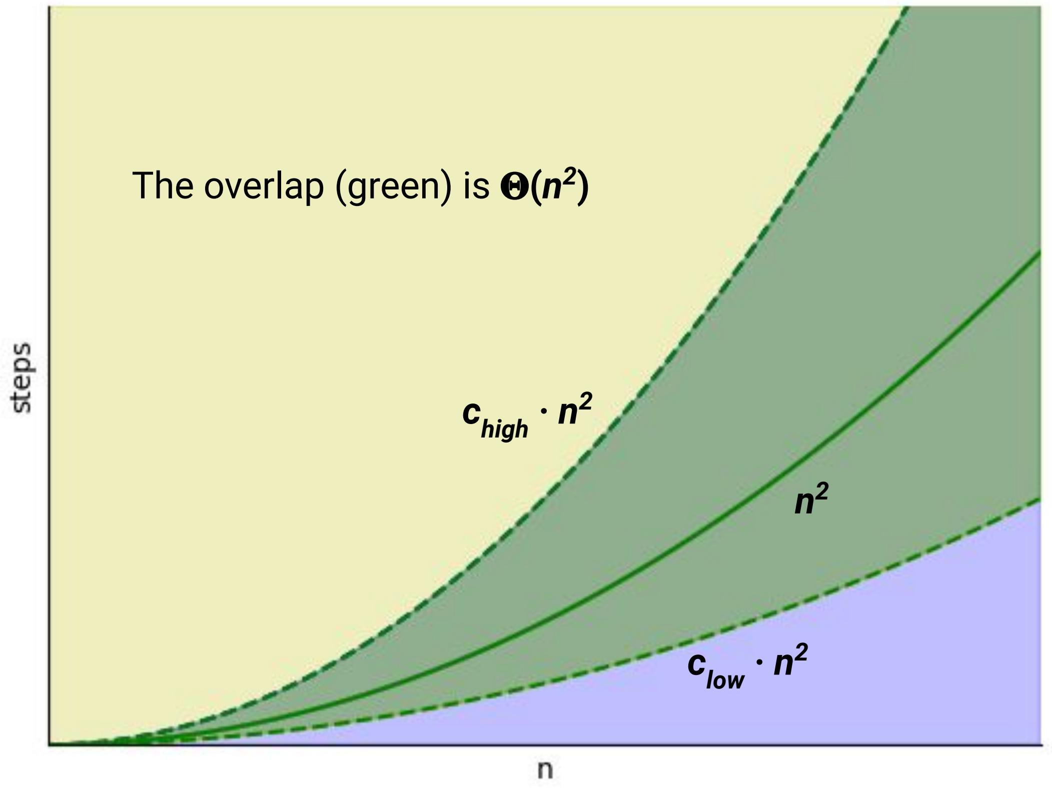

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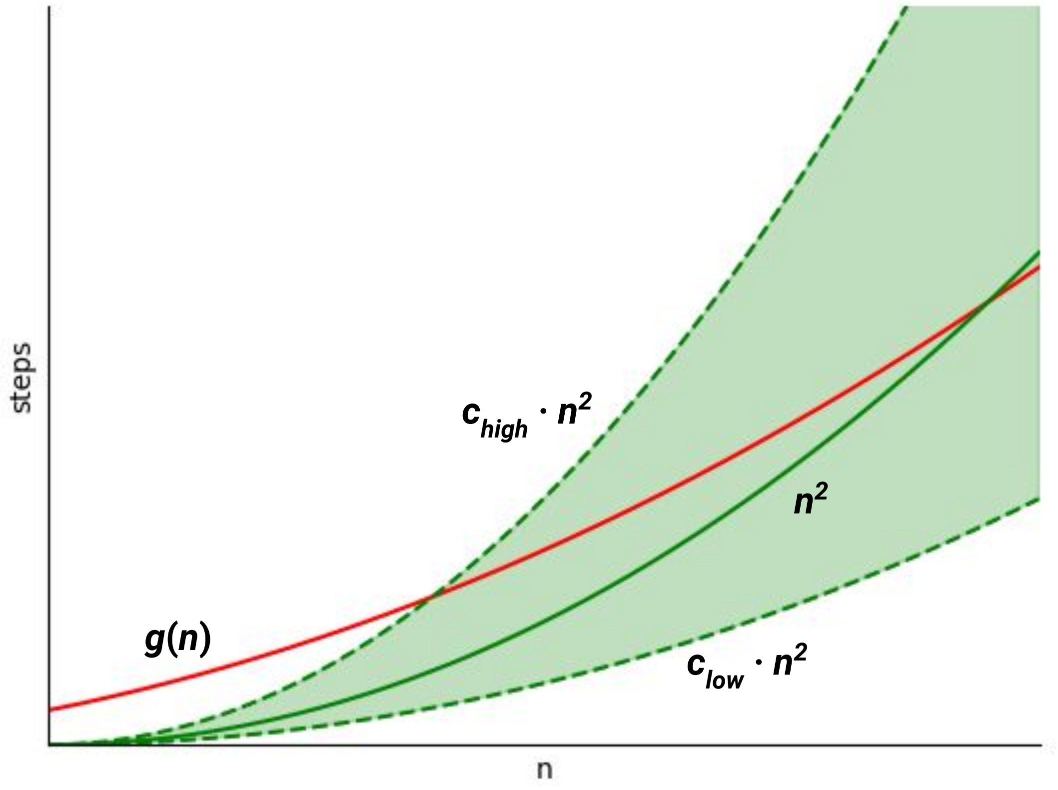
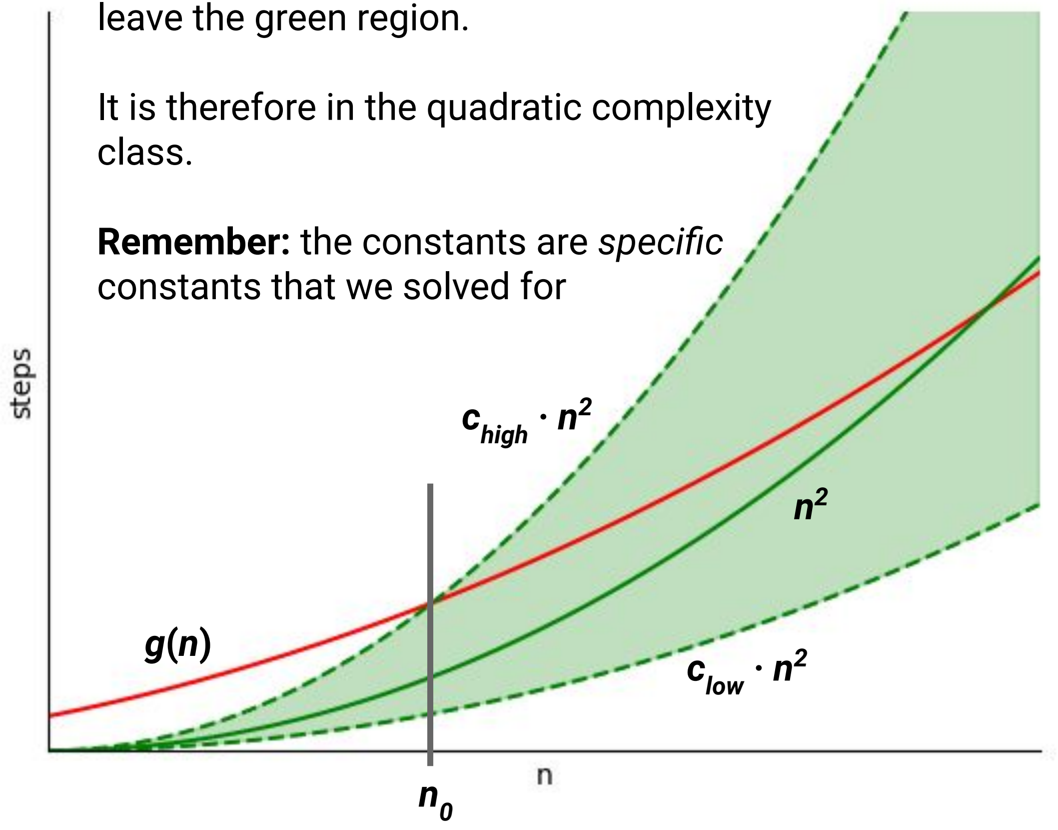


Complexity Class: Big Θ

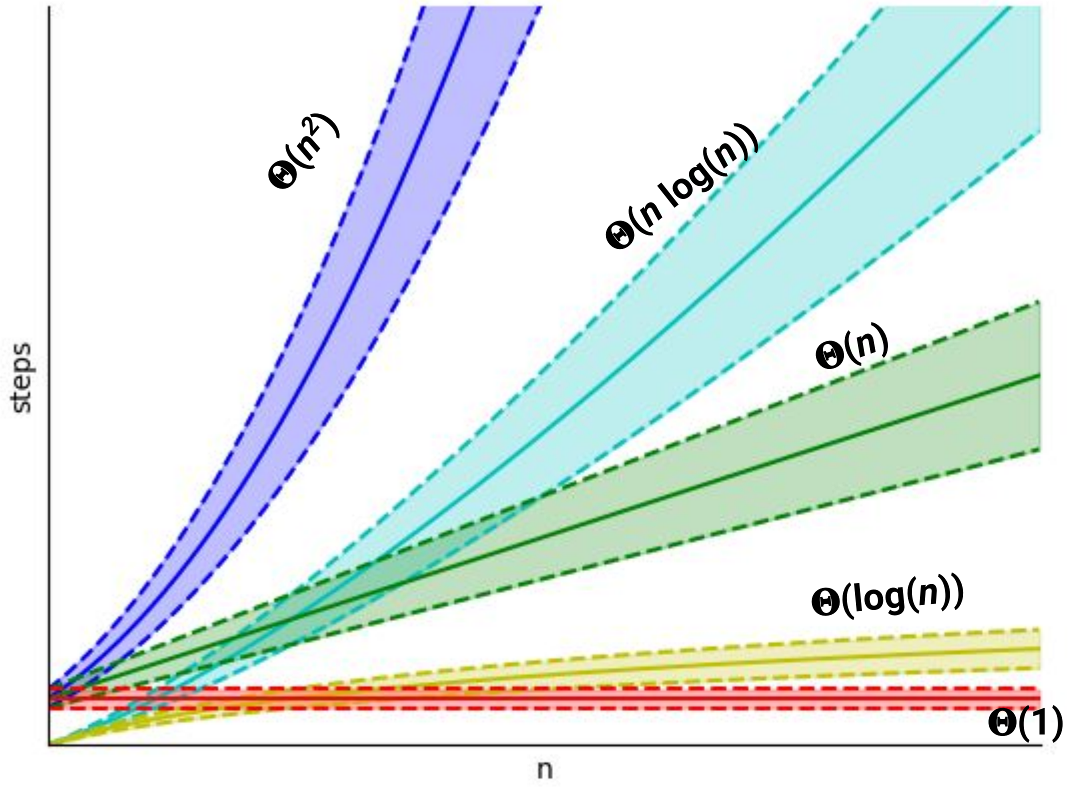
Once we move past n_0, $g(n)$ will never leave the green region.

It is therefore in the quadratic complexity class.

Remember: the constants are *specific* constants that we solved for

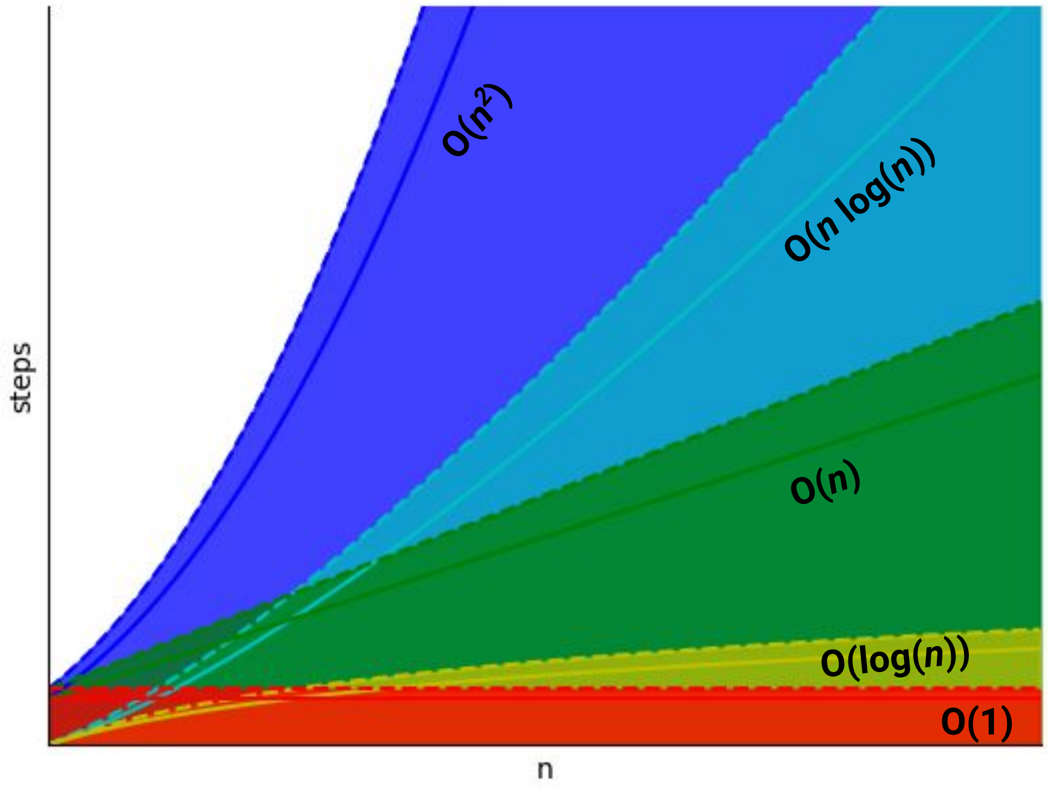


Complexity Class Ranking



$\Theta(1) < \Theta(\log(n)) < \Theta(n) < \Theta(n \log(n)) < \Theta(n^2) < \Theta(n^3) < \Theta(2^n)$

Big O Subsets



$O(1) \subset O(\log(n)) \subset O(n) \subset O(n \log(n)) \subset O(n^2) \subset O(n^3) \subset O(2^n)$

Shortcut

What complexity class do each of the following belong to:

$f(n) = 4n + n^2$

$g(n) = 2^n + 4n$

$h(n) = 100\, n \log(n) + 73n$

Shortcut

What complexity class do each of the following belong to:

$f(n) = 4n + n^2 \in \Theta(n^2)$

$g(n) = 2^n + 4n \in \Theta(2^n)$

$h(n) = 100\ n \log(n) + 73n \in \Theta(n \log(n))$

Shortcut

What complexity class do each of the following belong to:

$f(n) = 4n + n^2 \in \Theta(n^2)$

$g(n) = 2^n + 4n \in \Theta(2^n)$

$h(n) = 100\, n \log(n) + 73n \in \Theta(n \log(n))$

Shortcut: Just consider the complexity of the most dominant term

Why Focus on Dominating Terms?

$f(n)$	10	20	50	100	1000
$log(log(n))$	0.43 ns	0.52 ns	0.62 ns	0.68 ns	0.82 ns
$log(n)$	0.83 ns	1.01 ns	1.41 ns	1.66 ns	2.49 ns
n	2.5 ns	5 ns	12.5 ns	25 ns	0.25 µs
$nlog(n)$	8.3 ns	22 ns	71 ns	0.17 µs	2.49 µs
n^2	25 ns	0.1 µs	0.63 µs	2.5 µs	0.25 ms
n^5	25 µs	0.8 ms	78 ms	2.5 s	**2.9 days**
2^n	0.25 µs	0.26 ms	**3.26 days**	**10^{13} years**	**10^{284} years**
$n!$	0.91 ms	**19 years**	**10^{47} years**	**10^{141} years**	🤯

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

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Is $f(n)$ in $O(n^3)$?

Tight Bounds

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Is $f(n)$ in $O(n^3)$? ✓

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$?

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$? ✓

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$? ✓

Is $f(n)$ in $O(n)$?

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$? ✓

Is $f(n)$ in $O(n)$? ✗

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in O(n^2)$ ✓

Is $f(n)$ in $O(n^3)$? ✓

Is $f(n)$ in $O(2^n)$? ✓

Is $f(n)$ in $O(n)$? ✗

n^2, n^3, and 2^n all bound $f(n)$ from above

n^2 is a **tight** upper bound of $f(n)$

(there is no smaller upper bound for $f(n)$)

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$?

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$?

Tight Bounds

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Is $f(n)$ in $\Omega(\log(n))$? ✓

Is $f(n)$ in $\Omega(n)$?

Tight Bounds

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Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$? ✓

Is $f(n)$ in $\Omega(n)$? ✓

Tight Bounds

$f(n) = 4n + n^2 \in \Theta(n^2)$, therefore $f(n) = 4n + n^2 \in \Omega(n^2)$ ✓

Is $f(n)$ in $\Omega(n^3)$? ✗

Is $f(n)$ in $\Omega(\log(n))$? ✓

Is $f(n)$ in $\Omega(n)$? ✓

n^2, n, and $\log(n)$ all bound $f(n)$ from below
n^2 is a **tight** lower bound of $f(n)$
(there is no larger lower bound for $f(n)$)

Tight Bounds

If $g(n) \in \Theta(f(n))$, then:

- $g(n) \in O(f(n))$ is a tight upper bound
- $g(n) \in \Omega(f(n))$ is a tight lower bound

Tight Bounds

If $g(n) \in \Theta(f(n))$, then:

- $g(n) \in O(f(n))$ is a tight upper bound
- $g(n) \in \Omega(f(n))$ is a tight lower bound

But what if the tight upper bound and tight lower bound are not the same?

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function?

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $\boldsymbol{T(n) \in O(n^2)}$

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $\boldsymbol{T(n) \in O(n^2)}$

What is the tight lower bound of this function?

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $\boldsymbol{T(n) \in O(n^2)}$

What is the tight lower bound of this function? $\boldsymbol{T(n) \in \Omega(n)}$

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $\boldsymbol{T(n) \in O(n^2)}$

What is the tight lower bound of this function? $\boldsymbol{T(n) \in \Omega(n)}$

What is the complexity class of this function?

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

What is the tight upper bound of this function? $\boldsymbol{T(n) \in O(n^2)}$

What is the tight lower bound of this function? $\boldsymbol{T(n) \in \Omega(n)}$

What is the complexity class of this function? It does not have one!

Multi-class Functions

$$T(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

It is not bounded from above by n, therefore it cannot be in $\Theta(n)$

It is not bounded from below by n^2, therefore it cannot be in $\Theta(n^2)$

What is the tight upper bound of this function? $T(n) \in O(n^2)$

What is the tight lower bound of this function? $T(n) \in \Omega(n)$

What is the complexity class of this function? It does not have one!