CSE 250

Data Structures

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208 Capen Hall

Lec 20: Graph Traversals

Announcements

- PA2 released
  - Testing phase due Sunday 3/17
  - Implementation due Sunday 3/31
  - AutoLab open soon

So…what do we do with our graphs?

Connectivity Problems

Given graph ***G***:

Connectivity Problems

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- Is vertex ***u*** **adjacent** to vertex ***v***?

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- Is vertex ***u*** **adjacent** to vertex ***v***?
- Is vertex ***u*** **connected** to vertex ***v*** via some path?

Connectivity Problems

Given graph ***G***:

- Is vertex ***u*** **adjacent** to vertex ***v***?
- Is vertex ***u*** **connected** to vertex ***v*** via some path?
- Which vertices are **connected** to vertex ***v***?

Connectivity Problems

Given graph ***G***:

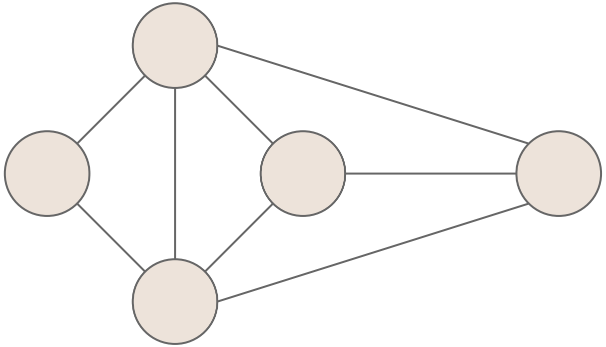
- Is vertex ***u*** **adjacent** to vertex ***v***?
- Is vertex ***u*** **connected** to vertex ***v*** via some path?
- Which vertices are **connected** to vertex ***v***?
- What is the **shortest path** from vertex ***u*** to vertex ***v***?

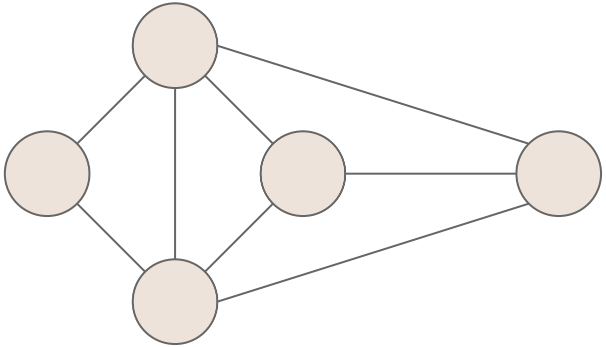
A few more definitions

A **subgraph**, ***S***, of a graph ***G*** is a graph where:

S's vertices are a subset of ***G***'s vertices

S's edges are a subset of ***G***'s edges





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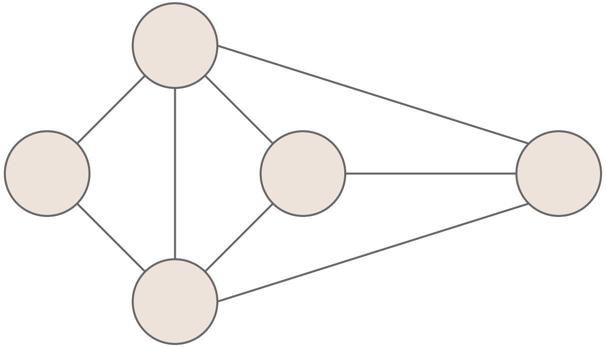
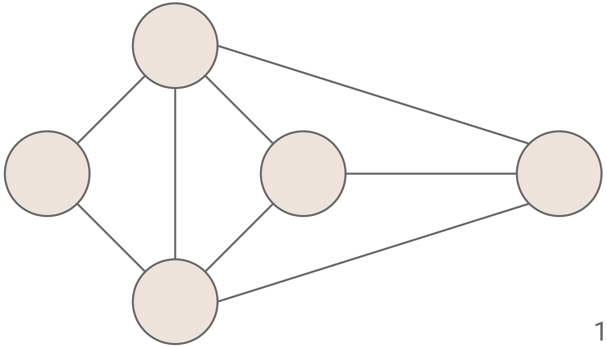


A **spanning subgraph** of ***G*...**

Is a subgraph of ***G***

Contains all of ***G***'s vertices



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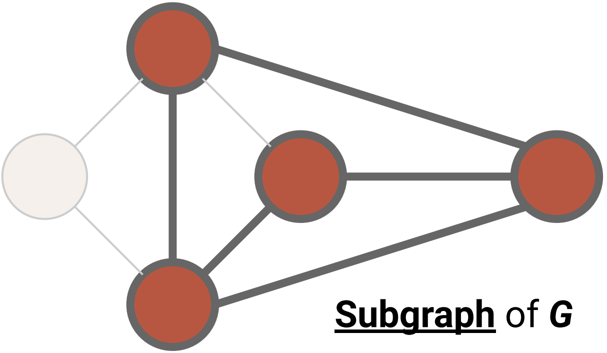

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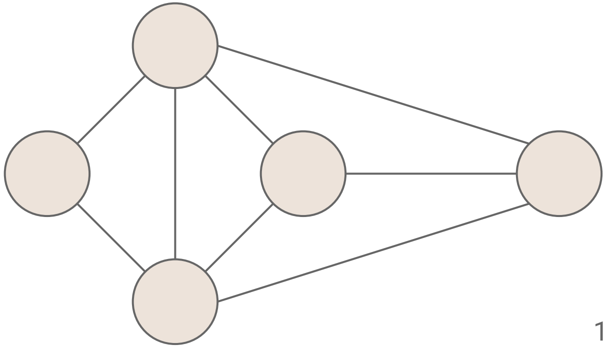

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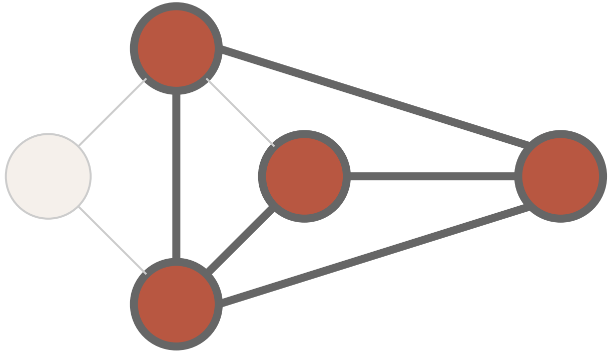

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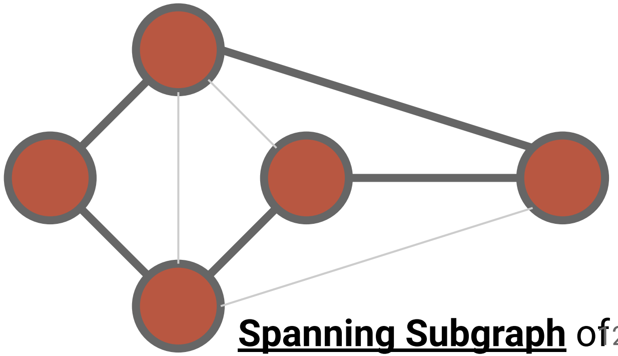
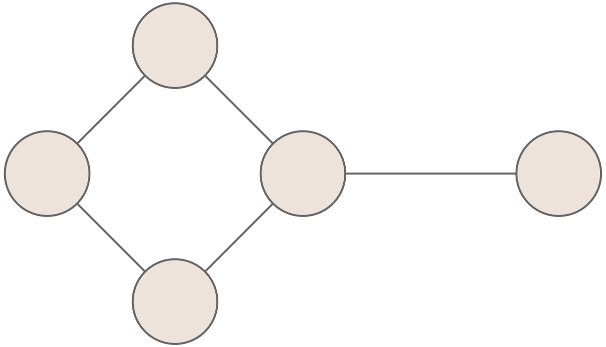


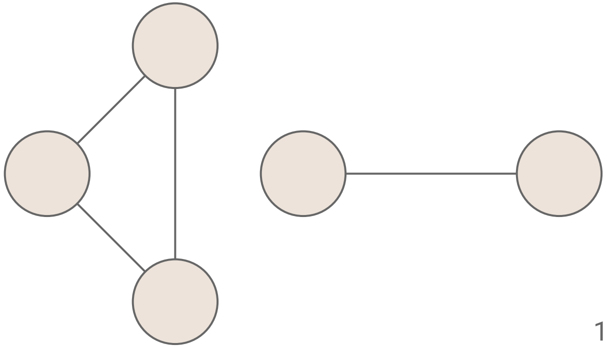
Spanning Subgraph of ***G***

A few more definitions

A graph is **connected**…

If there is a path between every pair of vertices





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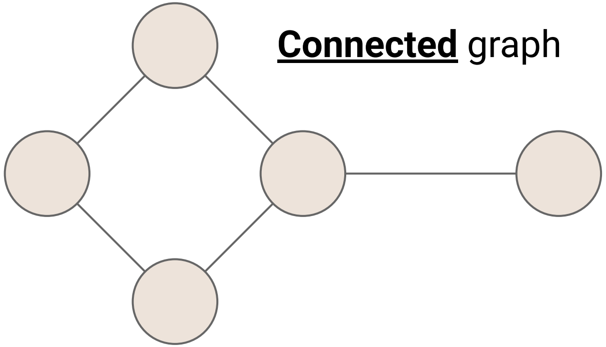

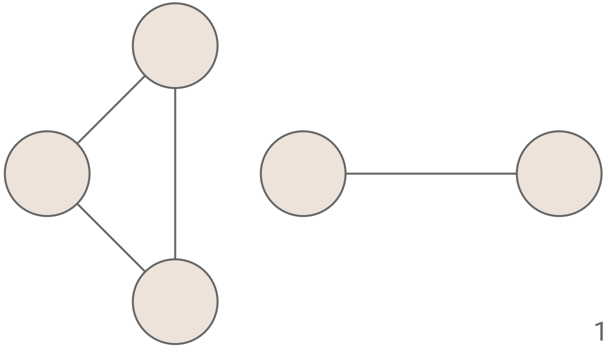

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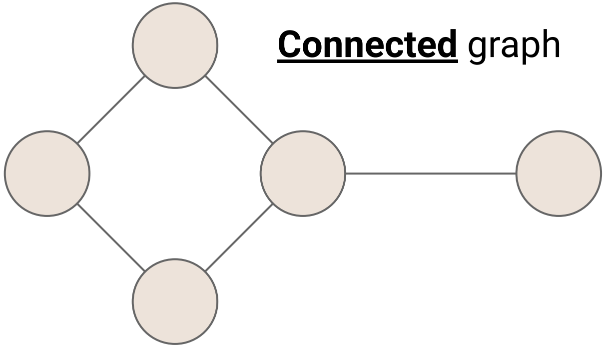
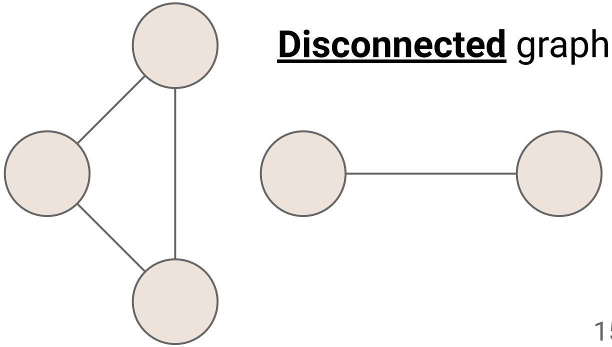


Connected graph



Disconnected graph

A few more definitions

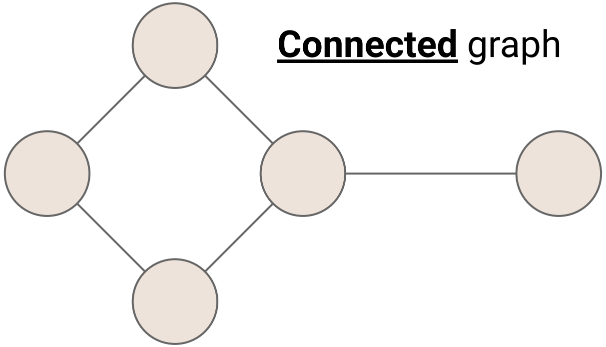
A graph is **connected**…

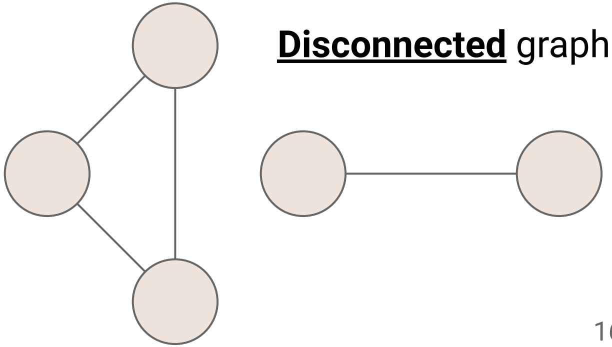
If there is a path between every pair of vertices

A **connected component** of ***G***…

Is a maximal connected subgraph of ***G***

- "maximal" means you can't add a new vertex without breaking the property
- Any subset of ***G***'s edges that connect the subgraph are fine





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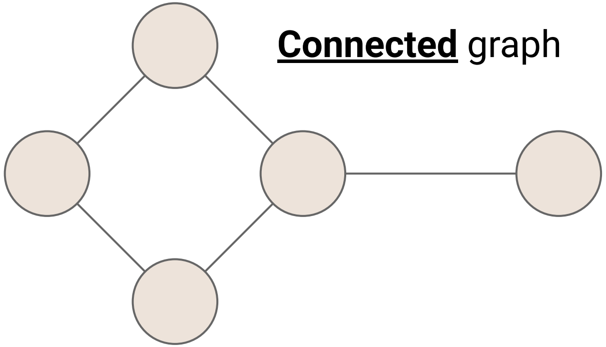

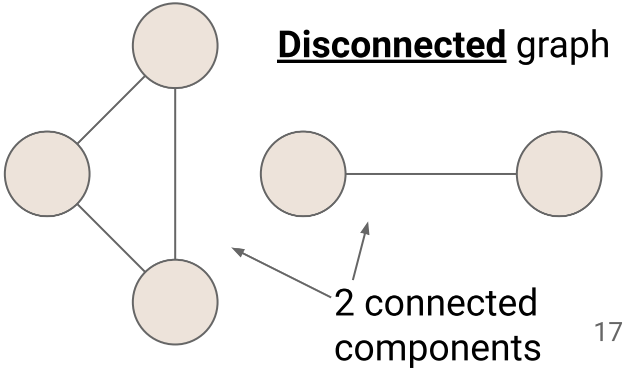

A few more definitions

A **free tree** is an undirected graph ***T*** such that…

There is exactly one simple path between any two nodes

- ***T*** is connected
- ***T*** has no cycles

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One vertex of ***T*** is the **root**

There is exactly one simple path from the root to every other vertex in the graph

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A **rooted tree** is a directed graph ***T*** such that…

One vertex of ***T*** is the **root**

There is exactly one simple path from the root to every other vertex in the graph

A (free/rooted) **forest** is a graph ***F*** such that…

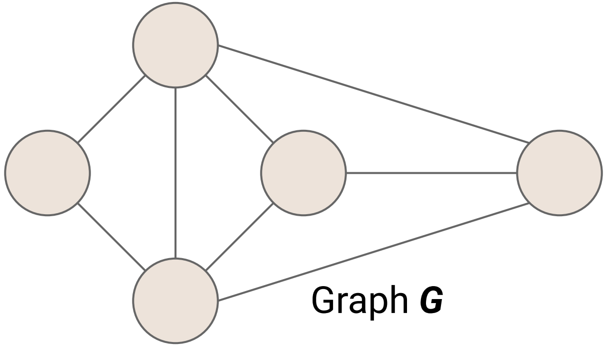
Every connected component is a tree

A few more definitions

A **spanning tree** of a connected graph…

…Is a spanning subgraph that is a tree

…It is not unique unless the graph is a tree



Graph ***G***

A few more definitions

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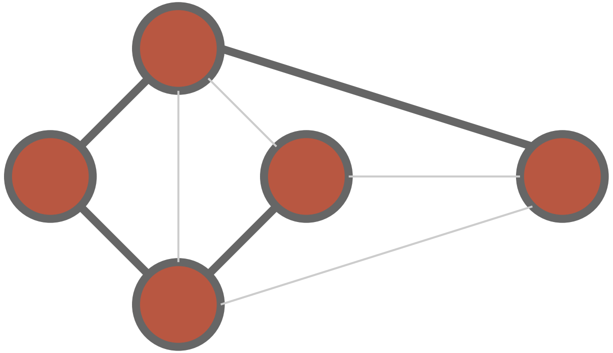
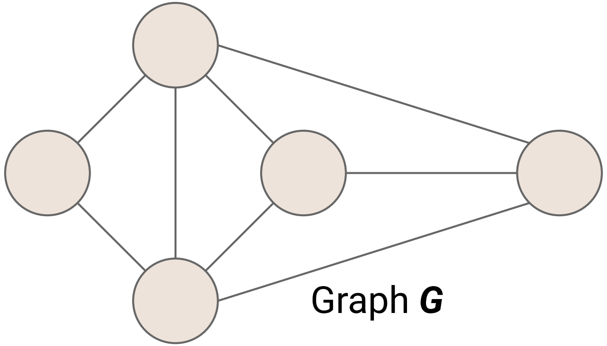
A **Spanning Tree** of ***G***





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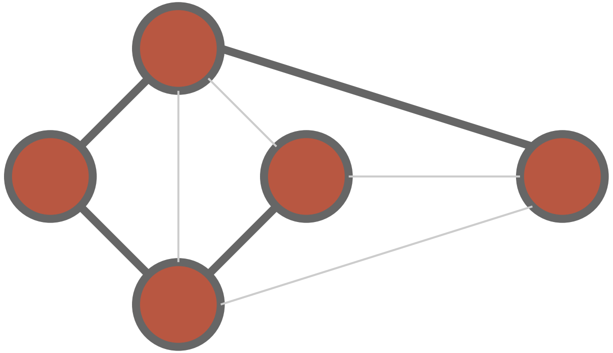
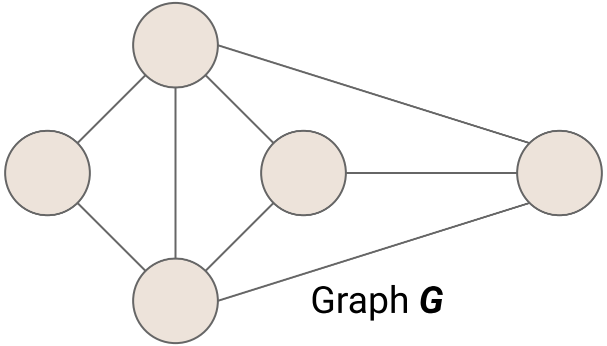
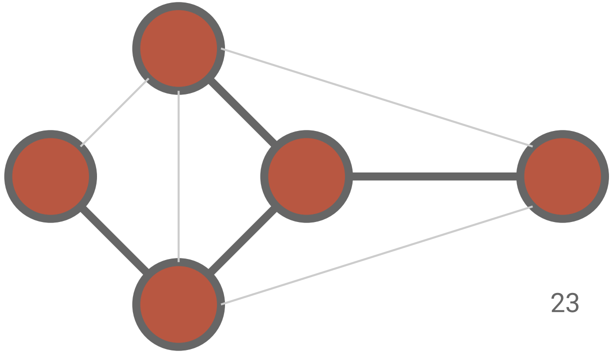




Graph ***G***

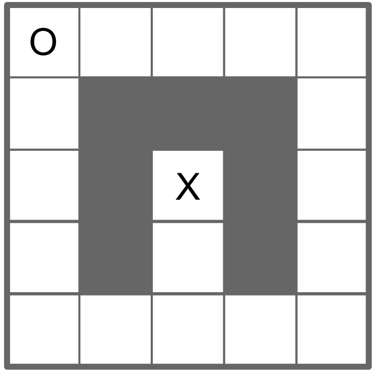
Another **Spanning Tree** of ***G***



Now back to the question…Connectivity

Back to Mazes

How could we represent our maze as a graph?



Back to Mazes

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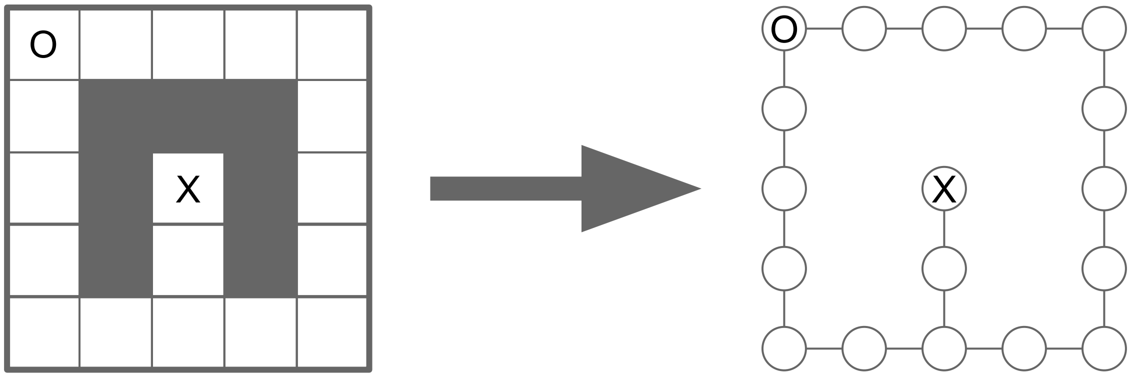

Recall

Searching the maze with a stack

We try every path, one at a time, following it as far as we can

…then backtrack and try another

Recall

Searching the maze with a stack (Depth-First Search)

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Searching with a queue?

TBD…

Depth-First Search

Primary Goals

- Visit every vertex in graph ***G* = (*V*,*E*)**
- Construct a spanning tree for every connected component

Depth-First Search

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  - **Side Effect:** Compute connected components

Depth-First Search

Primary Goals

- Visit every vertex in graph ***G* = (*V*,*E*)**
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  - **Side Effect:** Compute a path between all connected vertices

Depth-First Search

Primary Goals

- Visit every vertex in graph ***G* = (*V*,*E*)**
- Construct a spanning tree for every connected component
  - **Side Effect:** Compute connected components
  - **Side Effect:** Compute a path between all connected vertices
  - **Side Effect:** Determine if the graph is connected

Depth-First Search

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- Visit every vertex in graph ***G* = (*V*,*E*)**
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  - **Side Effect:** Identify cycles

Depth-First Search

<u>Primary Goals</u>

- Visit every vertex in graph ***G* = (*V*,*E*)**
- Construct a spanning tree for every connected component
    - **Side Effect:** Compute connected components
    - **Side Effect:** Compute a path between all connected vertices
    - **Side Effect:** Determine if the graph is connected
    - **Side Effect:** Identify cycles
- Complete in time $O(|V| + |E|)$

Depth-First Search

DFS

Input: Graph ***G* = (*V*,*E*)**

Output: Label every edge as:

- Spanning Edge: Part of the spanning tree
- Back Edge: Part of a cycle

Depth-First Search

DFS

Input: Graph $G = (V,E)$

Output: Label every edge as:

- Spanning Edge: Part of the spanning tree
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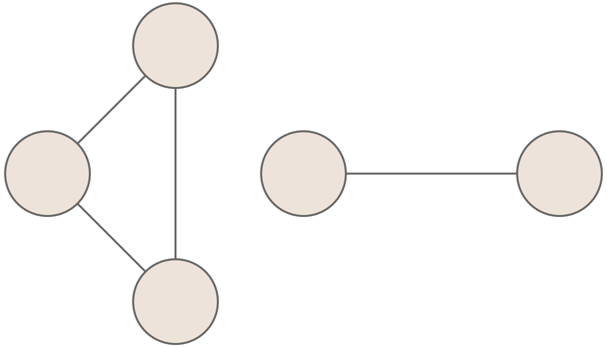
DFSOne

Input: Graph $G = (V,E)$, start vertex $v \in V$

Output: Label every edge in v's connected component

Depth-First Search



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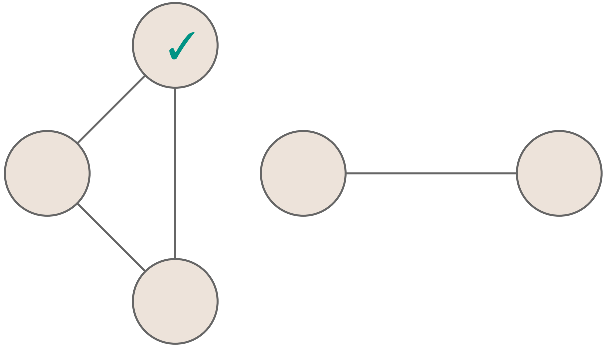

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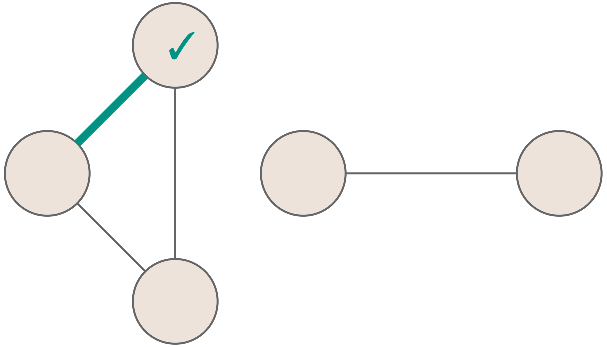

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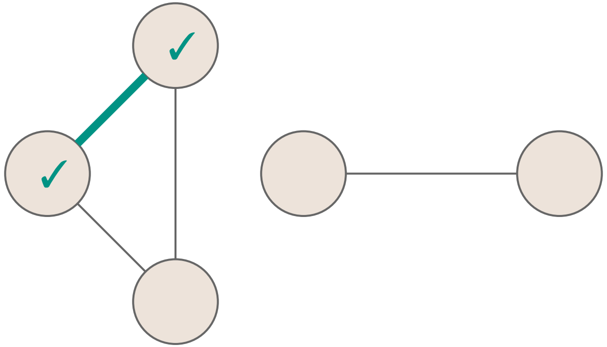

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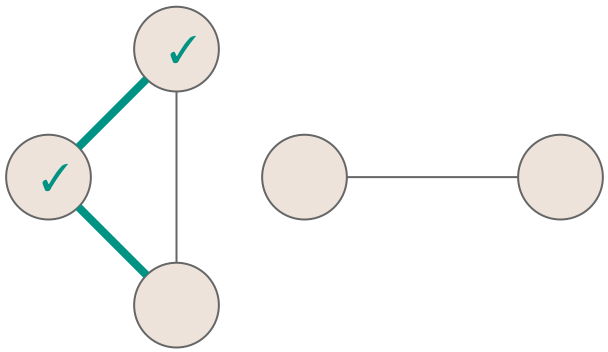

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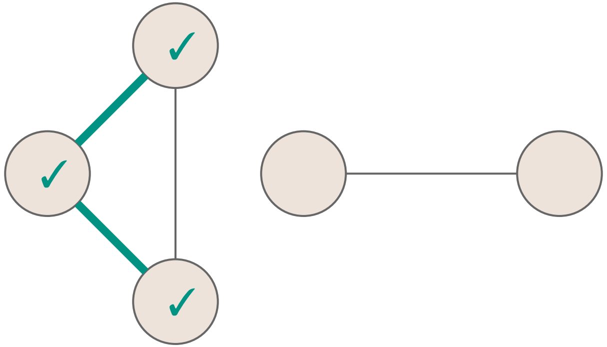

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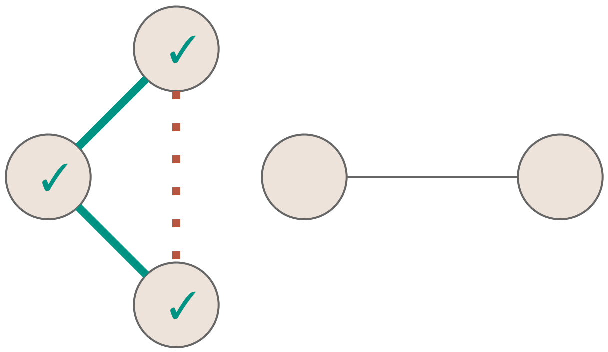

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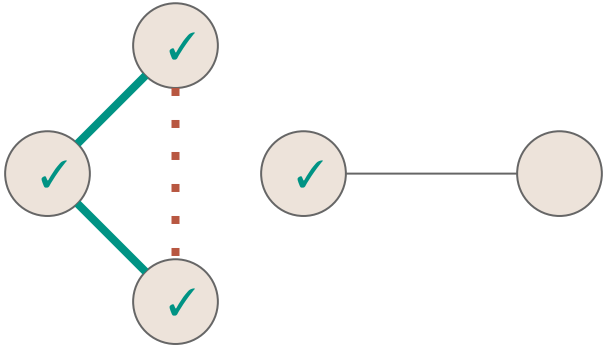

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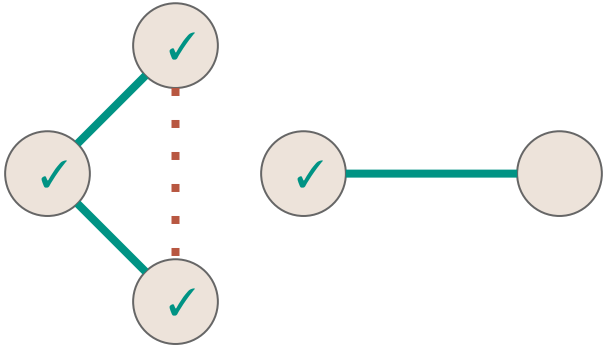

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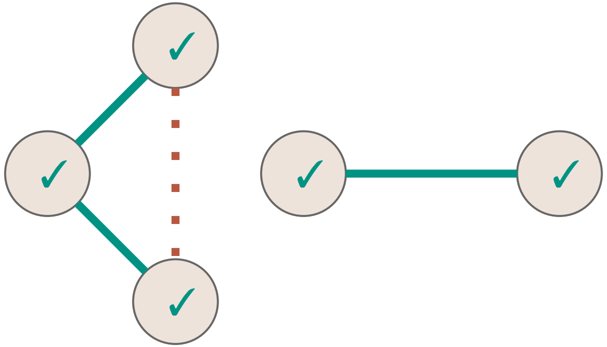

DFS

```
public void DFS(Graph graph) {
  for (Vertex v : graph.vertices) {
    v.setLabel(UNEXPLORED);
  }
  for (Edge e : graph.edges) {
    e.setLabel(UNEXPLORED);
  }
  for (Vertex v : graph.vertices) {
    if (v.label == UNEXPLORED) {
      DFSOne(graph, v);
    }
  }
}
```

DFS

```java
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  }
}
```

Initialize all vertices and edges to UNEXPLORED

DFS

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      DFSOne(graph, v);
    }
  }
}
```

Call DFSOne to label the connected component of every unexplored vertex

DFSOne

```
public void DFSOne(Graph graph, Vertex v) {
  v.setLabel(VISITED);
  for (Edge e : v.outEdges) {
    if (e.label == UNEXPLORED) {
      Vertex w = e.to;
      if (w.label == UNEXPLORED) {
        e.setLabel(SPANNING);
        DFSOne(graph, w);
      } else {
        e.setLabel(BACK);
      }
    }
}}
```

DFSOne

```
public void DFSOne(Graph graph, Vertex v) {
  v.setLabel(VISITED); ← Mark the vertex as VISITED (so we'll never try to visit it again)
  for (Edge e : v.outEdges) {
    if (e.label == UNEXPLORED) {
      Vertex w = e.to;
      if (w.label == UNEXPLORED) {
        e.setLabel(SPANNING);
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        e.setLabel(BACK);
      }
    }
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    }
}}
```

Check every outgoing edge (every possible way we could leave the current vertex)

DFSOne

```
public void DFSOne(Graph graph, Vertex v) {
  v.setLabel(VISITED);
  for (Edge e : v.outEdges) {
    if (e.label == UNEXPLORED) {
      Vertex w = e.to;
      if (w.label == UNEXPLORED) {
        e.setLabel(SPANNING);
        DFSOne(graph, w);
      } else {
        e.setLabel(BACK);
      }
    }
}}
```

Follow the unexplored edges

DFSOne

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public void DFSOne(Graph graph, Vertex v) {
  v.setLabel(VISITED);
  for (Edge e : v.outEdges) {
    if (e.label == UNEXPLORED) {
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      if (w.label == UNEXPLORED) {
        e.setLabel(SPANNING);
        DFSOne(graph, w);
      } else {
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    }
}}
```

If it leads to an unexplored vertex, then it is a spanning edge. Recursively explore that vertex.

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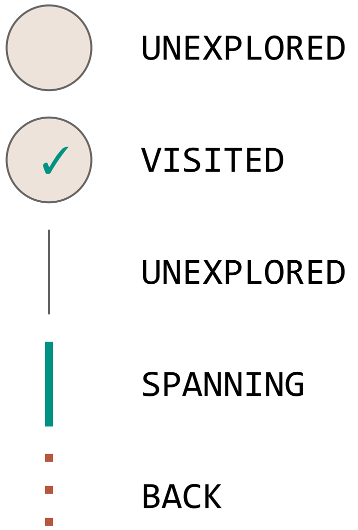
      } else {
        e.setLabel(BACK);
      }
    }
}}
```

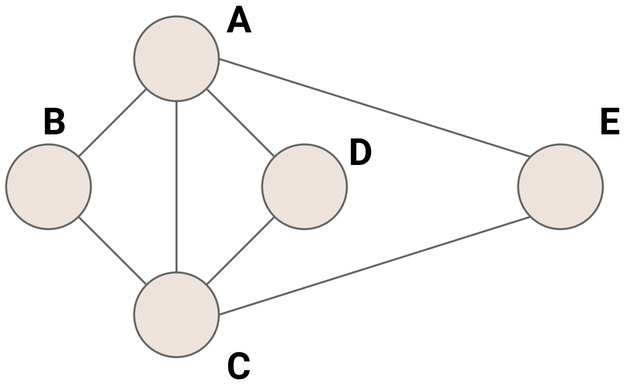
Otherwise, we just found a cycle

Detailed Example

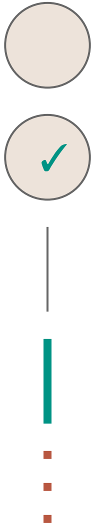


Call Stack

(→ edges to list)



Detailed Example



UNEXPLORED

VISITED

UNEXPLORED

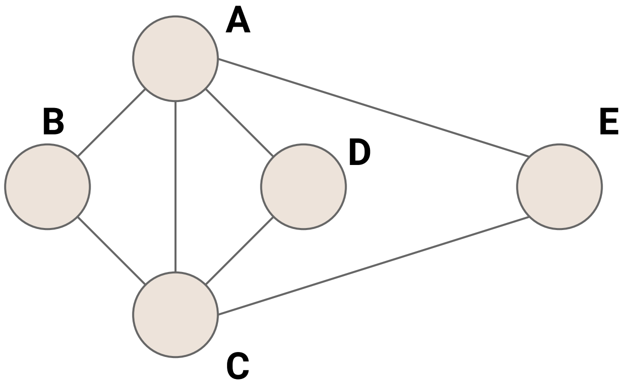
SPANNING

BACK

<u>Call Stack</u>

DFS(G)

[(→ edges to list)]



Detailed Example

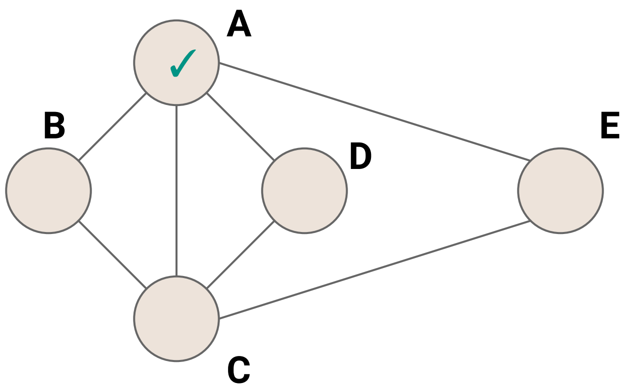
UNEXPLORED

✓ VISITED

UNEXPLORED

SPANNING

BACK



<u>Call Stack</u>
DFS(G)
DFSOne(G,A)

<u>(→ edges to list)</u>

Detailed Example

UNEXPLORED

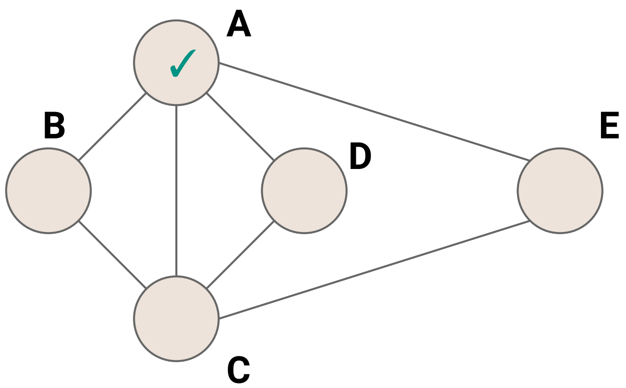
VISITED

UNEXPLORED

SPANNING

BACK

Call Stack | (→ edges to list)
DFS(G)
DFSOne(G,A) | (→ B, C, D, E)



Detailed Example

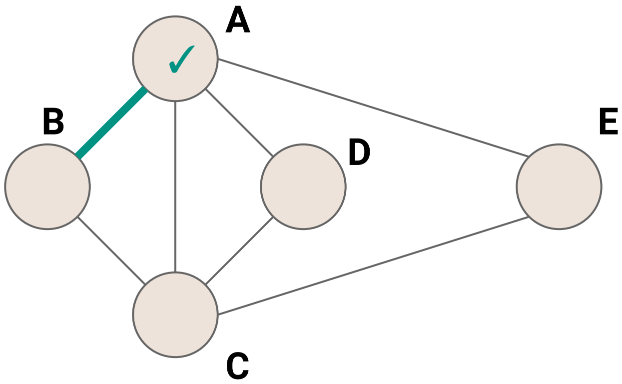
UNEXPLORED

✓ VISITED

UNEXPLORED

SPANNING

BACK



<u>Call Stack</u> <u>(→ edges to list)</u>

DFS(G)

DFSOne(G,A) (→ B, C, D, E)

Detailed Example

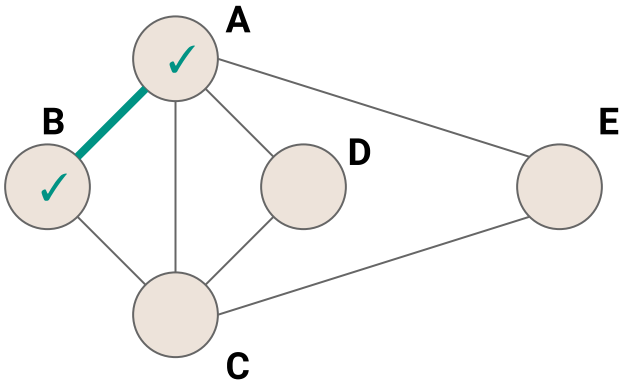
UNEXPLORED

VISITED

UNEXPLORED

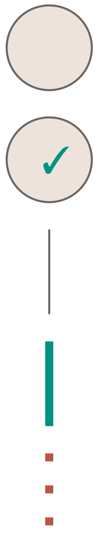
SPANNING

BACK



<u>Call Stack</u>	<u>(→ edges to list)</u>
`DFS(G)`	
`DFSOne(G,A)`	(→ B, C, D, E)
`DFSOne(G,B)`	(→ A, C)

Detailed Example



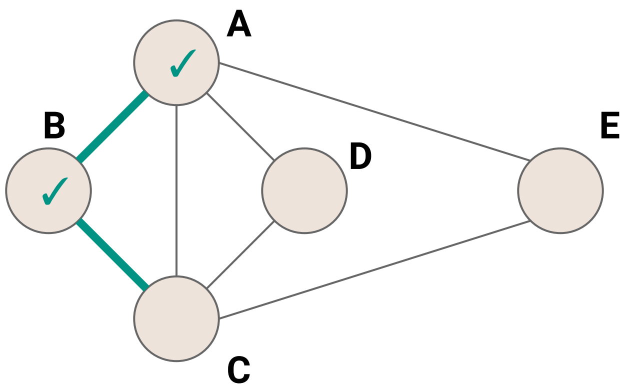
UNEXPLORED

VISITED

UNEXPLORED

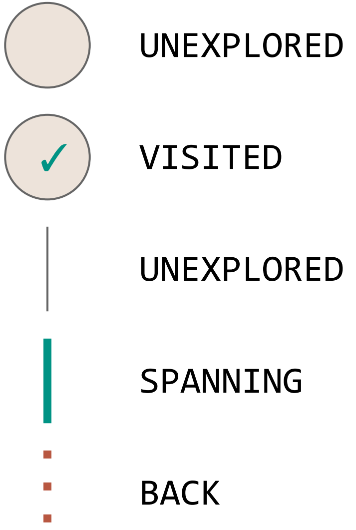
SPANNING

BACK

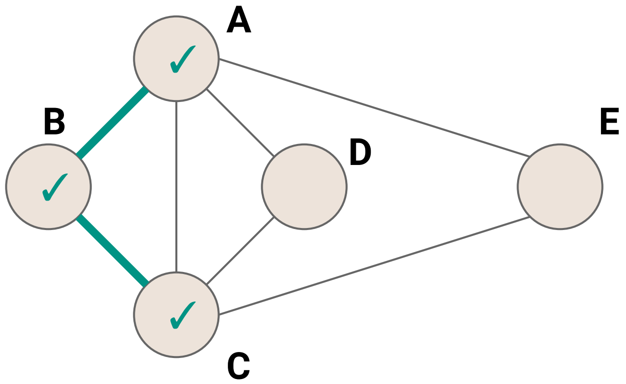


Call Stack	(→ edges to list)
DFS(G)	
DFSOne(G,A)	(→ B, C, D, E)
DFSOne(G,B)	(→ A, C)

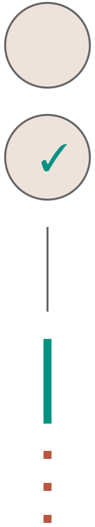
Detailed Example



Call Stack	(→ edges to list)
DFS(G)	
DFSOne(G,A)	(→ B, C, D, E)
DFSOne(G,B)	(→ A, C)
DFSOne(G,C)	(→ B, A, D, E)



Detailed Example



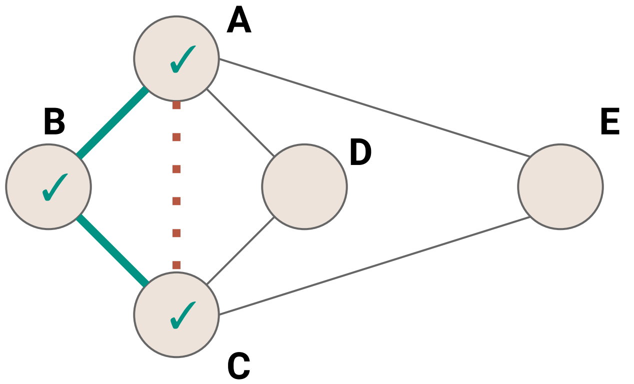
UNEXPLORED

VISITED

UNEXPLORED

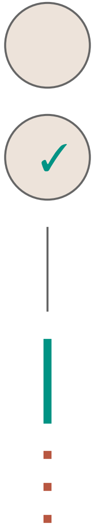
SPANNING

BACK



Call Stack	(→ edges to list)
`DFS(G)`	
`DFSOne(G,A)`	(→ B, C, D, E)
`DFSOne(G,B)`	(→ A, C)
`DFSOne(G,C)`	(→ B, A, D, E)

Detailed Example



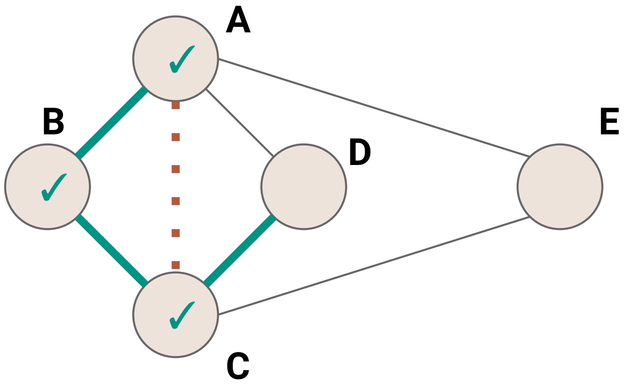
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

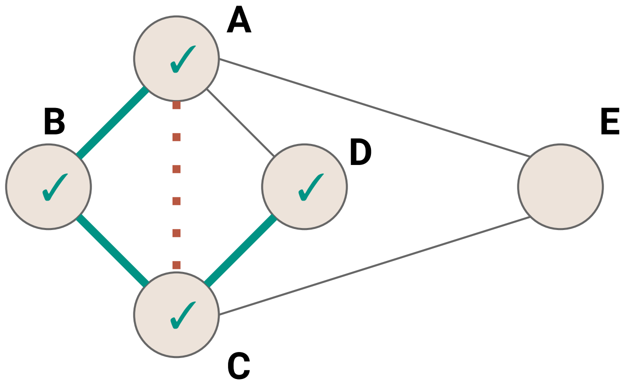
BACK



<u>Call Stack</u>	<u>(→ edges to list)</u>
DFS(G)	
DFSOne(G,A)	(→ B, C, D, E)
DFSOne(G,B)	(→ A, C)
DFSOne(G,C)	(→ B, A, D, E)

Detailed Example

- UNEXPLORED (node)
- VISITED (node, ✓)
- UNEXPLORED (edge)
- SPANNING (edge)
- BACK (edge)



<u>Call Stack</u>	<u>(→ edges to list)</u>
`DFS(G)`	
`DFSOne(G,A)`	(→ B, C, D, E)
`DFSOne(G,B)`	(→ A, C)
`DFSOne(G,C)`	(→ B, A, D, E)
`DFSOne(G,D)`	(→ A, C)

Detailed Example

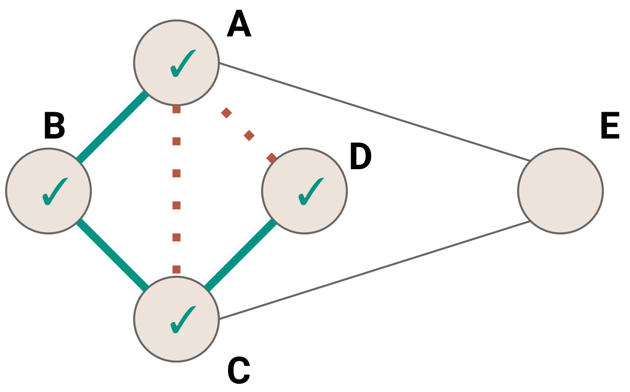
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

BACK



Call Stack	(→ edges to list)
DFS(G)	
DFSOne(G,A)	(→ B, C, D, E)
DFSOne(G,B)	(→ A, C)
DFSOne(G,C)	(→ B, A, D, E)
DFSOne(G,D)	(→ A, C)

Detailed Example

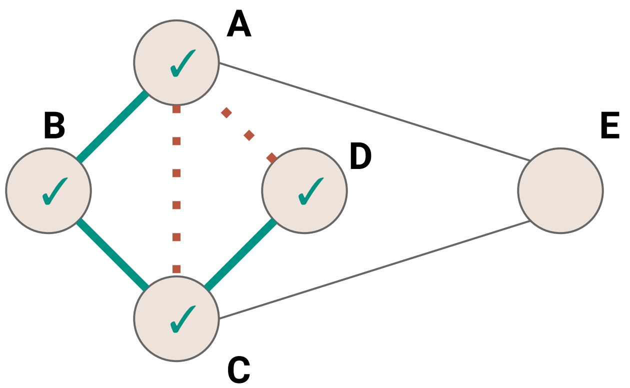
UNEXPLORED

VISITED

UNEXPLORED

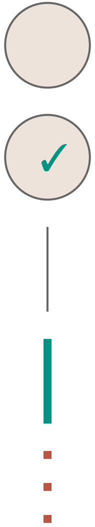
SPANNING

BACK



Call Stack	(→ edges to list)
DFS(G)	
DFSOne(G,A)	(→ B, C, D, E)
DFSOne(G,B)	(→ A, C)
DFSOne(G,C)	(→ B, A, D, E)

Detailed Example



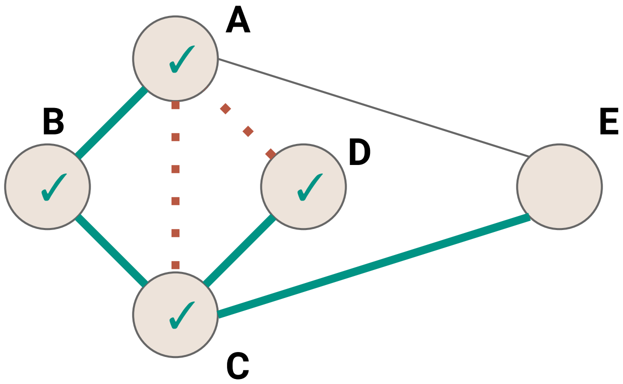
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

BACK



Call Stack	(→ edges to list)
`DFS(G)`	
`DFSOne(G,A)`	(→ B, C, D, E)
`DFSOne(G,B)`	(→ A, C)
`DFSOne(G,C)`	(→ B, A, D, E)

Detailed Example

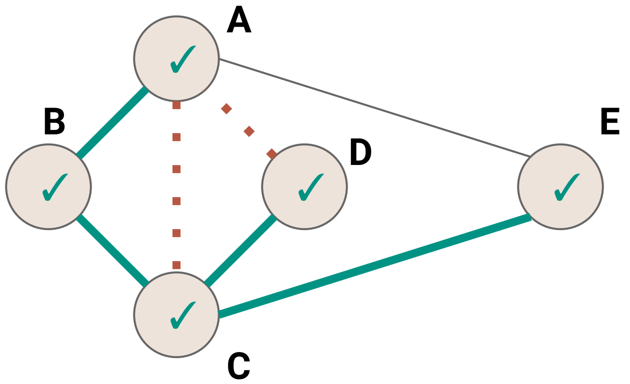
UNEXPLORED

✓ VISITED

UNEXPLORED

SPANNING

BACK



Call Stack	(→ edges to list)
DFS(G)	
DFSOne(G,A)	(→ B, C, D, E)
DFSOne(G,B)	(→ A, C)
DFSOne(G,C)	(→ B, A, D, E)
DFSOne(G,E)	(→ A, C)

Detailed Example

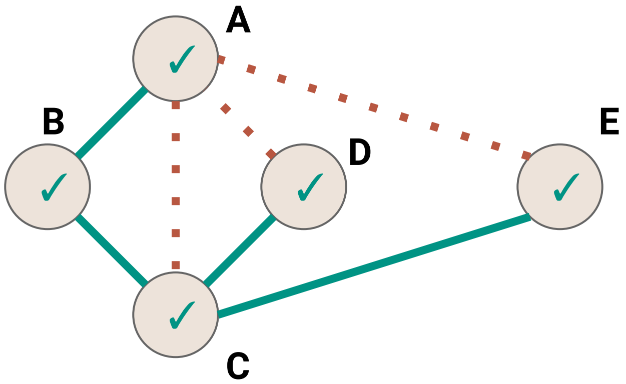
UNEXPLORED

✓ VISITED

UNEXPLORED

SPANNING

BACK



<u>Call Stack</u>	<u>(→ edges to list)</u>
DFS(G)	
DFSOne(G,A)	(→ B, C, D, E)
DFSOne(G,B)	(→ A, C)
DFSOne(G,C)	(→ B, A, D, E)
DFSOne(G,E)	(→ A, C)

Detailed Example

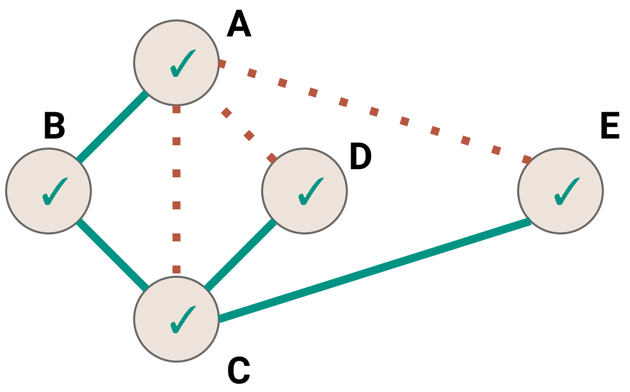
UNEXPLORED

✓ VISITED

UNEXPLORED

SPANNING

BACK



Call Stack	(→ edges to list)
DFS(G)	
DFSOne(G,A)	(→ B, C, D, E)
DFSOne(G,B)	(→ A, C)
DFSOne(G,C)	(→ B, A, D, E)

Detailed Example

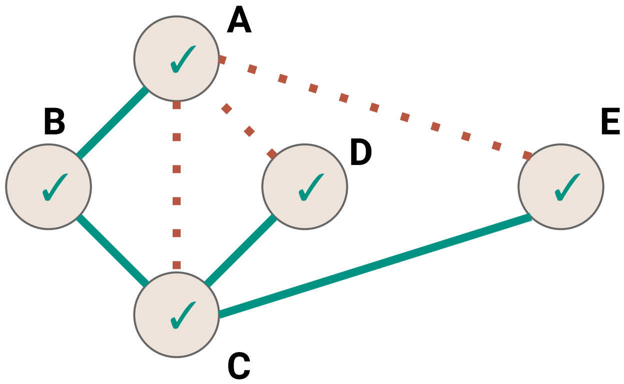
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

BACK



<u>Call Stack</u> <u>(→ edges to list)</u>

```
DFS(G)
DFSOne(G,A)     ( → B, C, D, E)
DFSOne(G,B)     ( → A, C)
```

Detailed Example

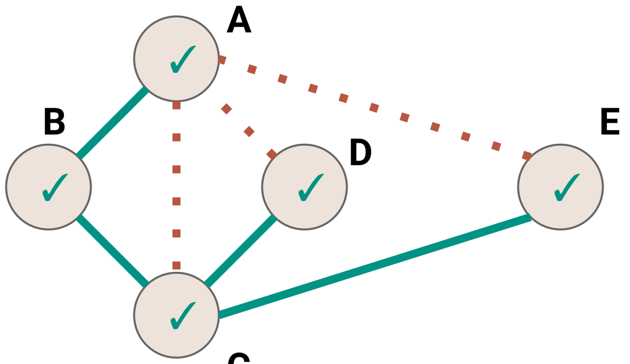
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

BACK



Call Stack	(→ edges to list)
DFS(G)	
DFSOne(G,A)	(→ B, C, D, E)

Detailed Example

UNEXPLORED

VISITED

UNEXPLORED

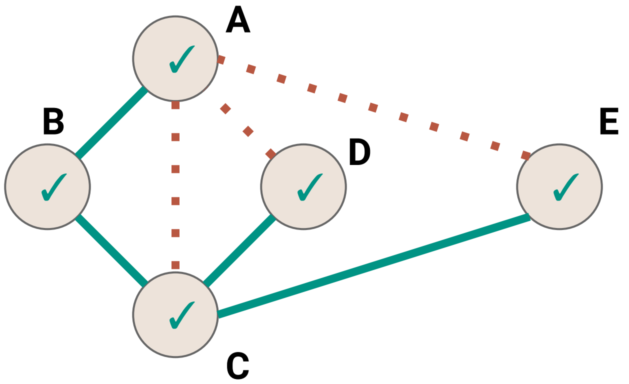
SPANNING

BACK

<u>Call Stack</u>

DFS(G)

(→ edges to list)



Detailed Example

UNEXPLORED

VISITED

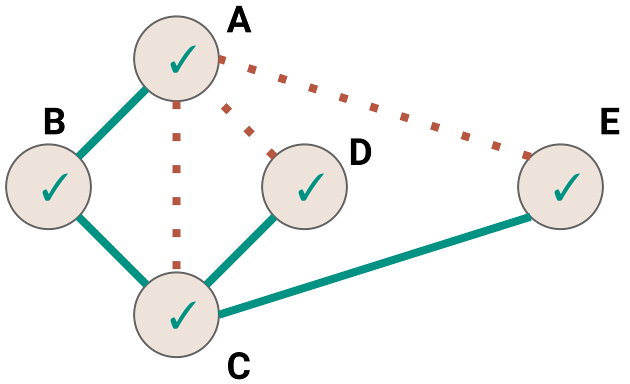
UNEXPLORED

SPANNING

BACK

Call Stack

(→ edges to list)



DFS vs Mazes

The DFS algorithm is like our stack-based maze solver (kind of)

- Mark each grid square with **VISITED** as we explore it
- Mark each path with **SPANNING** or **BACK**
- Only visit each vertex once (this differs from our maze search)

DFS vs Mazes

The DFS algorithm is like our stack-based maze solver (kind of)

- Mark each grid square with **VISITED** as we explore it
- Mark each path with **SPANNING** or **BACK**
- Only visit each vertex once (this differs from our maze search)
  - DFS will not necessarily find the shortest paths

Depth-First Search Complexity

What's the complexity?

DFS

```
public void DFS(Graph graph) {
  for (Vertex v : graph.vertices) {
    v.setLabel(UNEXPLORED);
  }
  for (Edge e : graph.edges) {
    e.setLabel(UNEXPLORED);
  }
  for (Vertex v : graph.vertices) {
    if (v.label == UNEXPLORED) {
      DFSOne(graph, v);
    }
  }
}
```

DFS

```
public void DFS(Graph graph) {
  Θ(|V|)
  for (Edge e : graph.edges) {
    e.setLabel(UNEXPLORED);
  }
  for (Vertex v : graph.vertices) {
    if (v.label == UNEXPLORED) {
      DFSOne(graph, v);
    }
  }
}
```

DFS

```
public void DFS(Graph graph) {
  Θ(|V|)
  Θ(|E|)
  for (Vertex v : graph.vertices) {
    if (v.label == UNEXPLORED) {
      DFSOne(graph, v);
    }
  }
}
```

DFS

```
public void DFS(Graph graph) {
  Θ(|V|)
  Θ(|E|)
  for (Vertex v : graph.vertices) {
    if (v.label == UNEXPLORED) {
      Θ(???)
    }
  }
}
```

DFSOne

```java
public void DFSOne(Graph graph, Vertex v) {
  v.setLabel(VISITED);
  for (Edge e : v.outEdges) {
    if (e.label == UNEXPLORED) {
      Vertex w = e.to;
      if (w.label == UNEXPLORED) {
        e.setLabel(SPANNING);
        DFSOne(graph, w);
      } else {
        e.setLabel(BACK);
      }
    }
}}
```

DFSOne

```
public void DFSOne(Graph graph, Vertex v) {
  Θ(1)
  for (Edge e : v.outEdges) {
    if (e.label == UNEXPLORED) {
      Θ(1)
      if (w.label == UNEXPLORED) {
        Θ(1)
        Θ(???)
      } else {
        Θ(1)
      }
    }
}}
```

Depth-First Search Complexity

How many times do we call **`DFSOne`** *on each vertex?*

Depth-First Search Complexity

How many times do we call **`DFSOne`** *on each vertex?*

Observation: **`DFSOne`** is called on each vertex *at most* once

If **`v.label == VISITED`**, both **`DFS`**, and **`DFSOne`** skip it

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$O(|V|)$ calls to **`DFSOne`**

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What's the runtime of **`DFSOne`** ***excluding the recursive calls?***

DFSOne

```
public void DFSOne(Graph graph, Vertex v) {
  Θ(1)
  for (Edge e : v.outEdges) {
    if (e.label == UNEXPLORED) {
      Θ(1)
      if (w.label == UNEXPLORED) {
        Θ(1)
        Θ(???)
      } else {
        Θ(1)
      }
    }
}}
```

DFSOne

```
public void DFSOne(Graph graph, Vertex v) {
  Θ(1)
  for (Edge e : v.outEdges) {
    Θ(1)
  }
}
```

DFSOne

```
public void DFSOne(Graph graph, Vertex v) {
  Θ(1)
  for (Edge e : v.outEdges) {
    Θ(1)
  }
}
```

As long as we use an adjacency list this will be able to iterate through the adjacenct edges in Θ (deg(v)) time

DFSOne

```
public void DFSOne(Graph graph, Vertex v) {
  Θ(1)
  Θ(deg(v))
}
```

Depth-First Search Complexity

How many times do we call **`DFSOne`** *on each vertex?*

Observation: **`DFSOne`** is called on each vertex *at most* once

If **`v.label == VISITED`**, both **`DFS`**, and **`DFSOne`** skip it

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How many times do we call **`DFSOne`** *on each vertex?*

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$O(|V|)$ calls to **`DFSOne`**

What's the runtime of **`DFSOne`** ***excluding the recursive calls?*** $O(\deg(v))$

Depth-First Search Complexity

What is the sum over all calls to **`DFSOne`**?

Depth-First Search Complexity

What is the sum over all calls to **`DFSOne`**?

$$\sum_{v \in V} O(deg(v))$$

Depth-First Search Complexity

What is the sum over all calls to **`DFSOne`**?

$$\sum_{v \in V} O(deg(v))$$

$$= O(\sum_{v \in V} deg(v))$$

Depth-First Search Complexity

What is the sum over all calls to **`DFSOne`**?

$$\sum_{v \in V} O(deg(v))$$

$$= O(\sum_{v \in V} deg(v))$$

$$= O(2|E|)$$

Depth-First Search Complexity

What is the sum over all calls to **`DFSOne`**?

$$\sum_{v \in V} O(deg(v))$$

$$= O(\sum_{v \in V} deg(v))$$

$$= O(2|E|)$$

$$= O(|E|)$$

Depth-First Search Complexity

In summary…

Depth-First Search Complexity

In summary…

1. Mark the vertices **`UNVISITED`**

Depth-First Search Complexity

In summary…

1. Mark the vertices **`UNVISITED`** $O(|V|)$

Depth-First Search Complexity

In summary…

1. Mark the vertices **`UNVISITED`** $O(|V|)$
2. Mark the edges **`UNVISITED`**

Depth-First Search Complexity

In summary…

1. Mark the vertices **`UNVISITED`** $O(|V|)$
2. Mark the edges **`UNVISITED`** $O(|E|)$

Depth-First Search Complexity

In summary…

1. Mark the vertices **`UNVISITED`** — $O(|V|)$
2. Mark the edges **`UNVISITED`** — $O(|E|)$
3. **`DFS`** vertex loop

Depth-First Search Complexity

In summary…

1. Mark the vertices **`UNVISITED`** — ***O***(|***V***|)
2. Mark the edges **`UNVISITED`** — ***O***(|***E***|)
3. **`DFS` vertex loop** — ***O***(|***V***|) **iterations**

Depth-First Search Complexity

In summary…

1. Mark the vertices **`UNVISITED`**	$O(\|V\|)$
2. Mark the edges **`UNVISITED`**	$O(\|E\|)$
3. **`DFS`** vertex loop	**$O(\|V\|)$ iterations**
4. All calls to **`DFSOne`**	

Depth-First Search Complexity

In summary…

1. Mark the vertices **`UNVISITED`**	$O(\|V\|)$
2. Mark the edges **`UNVISITED`**	$O(\|E\|)$
3. Sum of all calls to **`DFSOne`**	$O(\|E\|)$ **total**

Depth-First Search Complexity

In summary...

1. Mark the vertices `UNVISITED` — $O(|V|)$
2. Mark the edges `UNVISITED` — $O(|E|)$
3. Sum of all calls to `DFSOne` — $O(|E|)$ **total**

$O(|V| + |E|)$

DFS without Recursion

Our DFSOne implementation uses recursion for the search…

The recursive calls form a Stack…

Can we make a non-recursive implementation using a Stack explicitly?

```java
public void DFSOneNoRecursion(Graph graph, Vertex v) {
  Stack<Vertex> todo = new Stack<>();
  v.setLabel(VISITED);
  todo.push(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.pop();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          w.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.push(w);
        } else {
          e.setLabel(BACK);
        }
}}}}
```

```
public void DFSOneNoRecursion(Graph graph, Vertex v) {
  Stack<Vertex> todo = new Stack<>();
  v.setLabel(VISITED);
  todo.push(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.pop();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          w.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.push(w);
        } else {
          e.setLabel(BACK);
        }
}}}}
```

Use a stack to keep track of what vertices we want to visit (basically a running TODO list)

```java
public void DFSOneNoRecursion(Graph graph, Vertex v) {
  Stack<Vertex> todo = new Stack<>();
  v.setLabel(VISITED);
  todo.push(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.pop();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          w.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.push(w);
        } else {
          e.setLabel(BACK);
        }
}}}}
```

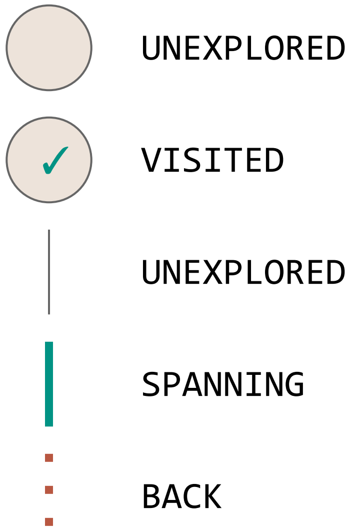
Pop a vertex from the Stack and check all of it's outgoing edges

```java
public void DFSOneNoRecursion(Graph graph, Vertex v) {
  Stack<Vertex> todo = new Stack<>();
  v.setLabel(VISITED);
  todo.push(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.pop();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          w.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.push(w);
        } else {
          e.setLabel(BACK);
        }
}}}}
```

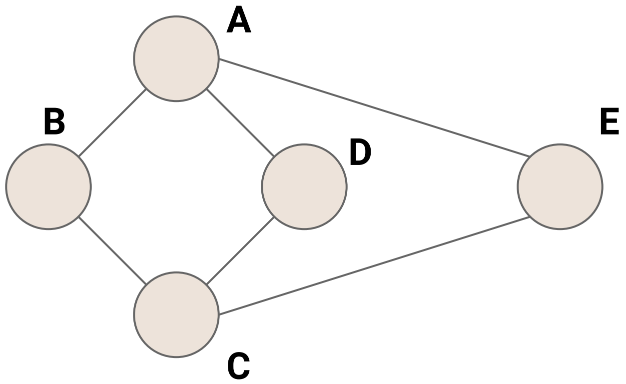
When we find a new vertex, mark it as VISITED, and add it to our TODO list.

Remember, our TODO list is a stack (LIFO) so whatever we push last will be the next thing we pop (and explore)

Detailed Example



TODO Stack



Detailed Example

UNEXPLORED

✓ VISITED

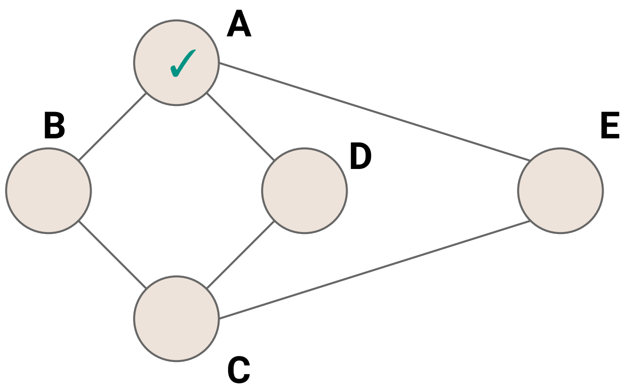
UNEXPLORED

SPANNING

BACK

<u>TODO Stack</u>

A



Detailed Example

UNEXPLORED

VISITED

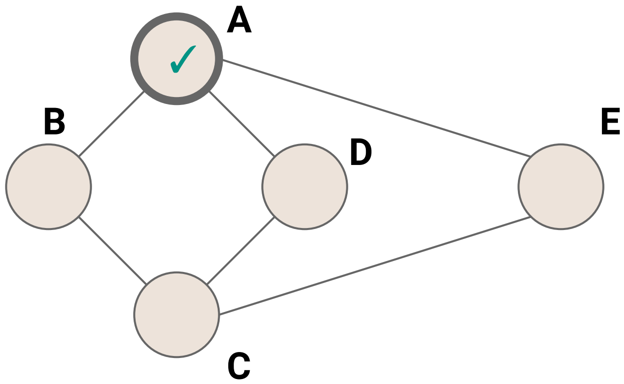
UNEXPLORED

SPANNING

BACK

Current Vertex: **A**

TODO Stack



Detailed Example

UNEXPLORED

VISITED

UNEXPLORED

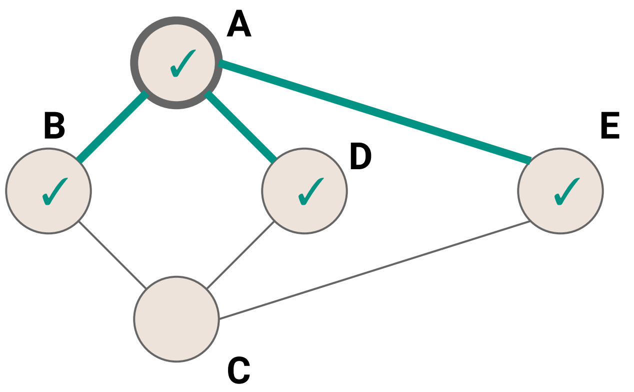
SPANNING

BACK

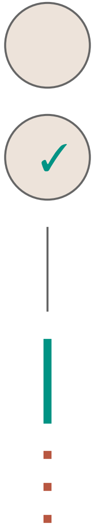
Current Vertex: **A**

<u>TODO Stack</u>

B
D
E



Detailed Example



UNEXPLORED

VISITED

UNEXPLORED

SPANNING

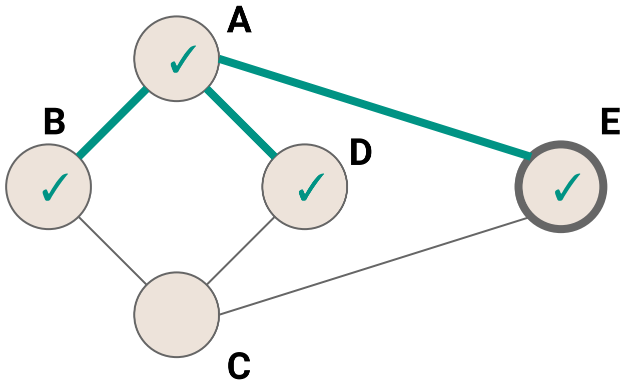
BACK

Current Vertex: **E**

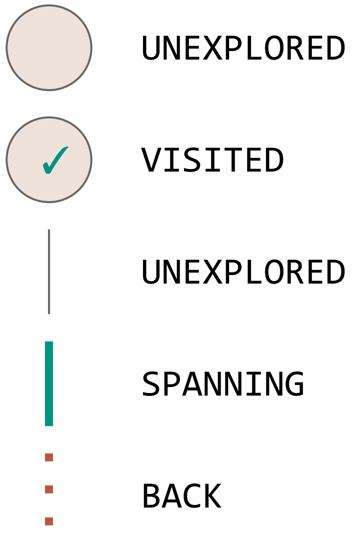
<u>TODO Stack</u>

B

D



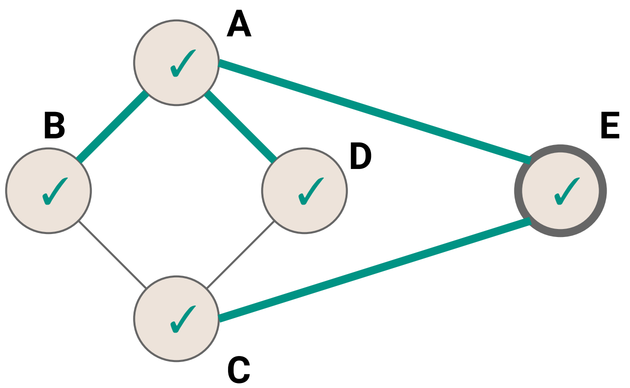
Detailed Example



Current Vertex: **E**

<u>TODO Stack</u>

B
D
C



Detailed Example

UNEXPLORED

VISITED

UNEXPLORED

SPANNING

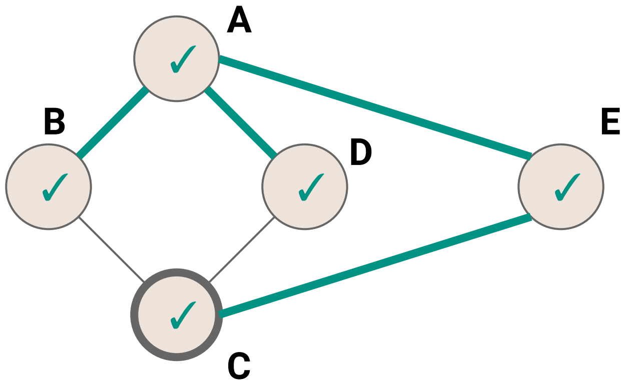
BACK

Current Vertex: **C**

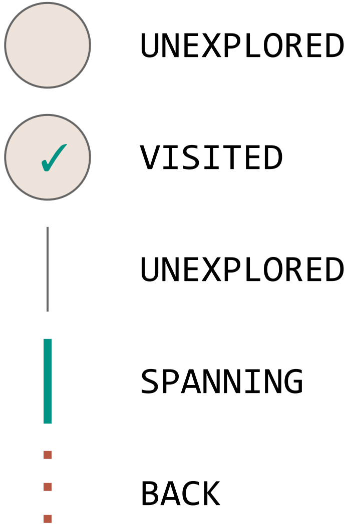
<u>TODO Stack</u>

B

D

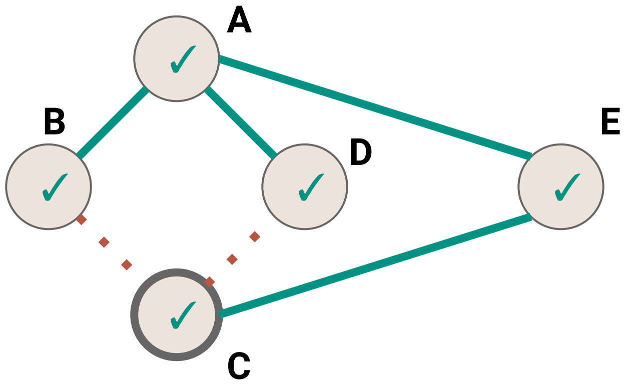


Detailed Example



Current Vertex: **C**

<u>TODO Stack</u>
B
D



Detailed Example

UNEXPLORED

VISITED

UNEXPLORED

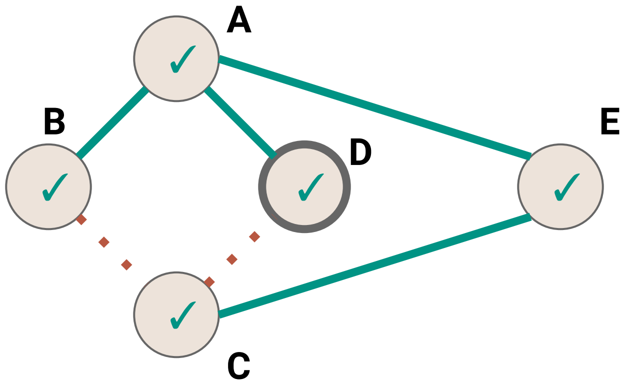
SPANNING

BACK

Current Vertex: **D**

<u>TODO Stack</u>

B



Detailed Example

UNEXPLORED

VISITED

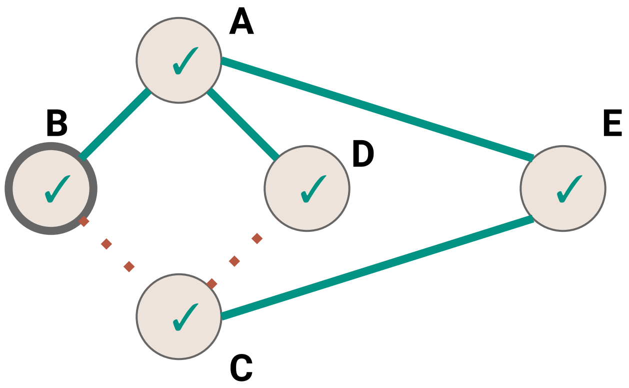
UNEXPLORED

SPANNING

BACK

Current Vertex: **B**

TODO Stack



```java
public void DFSOneNoRecursion(Graph graph, Vertex v) {
  Stack<Vertex> todo = new Stack<>();
  v.setLabel(VISITED);
  todo.push(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.pop();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == U
          curr.setLabel(
          e.setLabel(SPA
          todo.push(w);
        } else {
          e.setLabel(BACK);
        }
}}}}
```

Now back to our burning question…

What happens if we use a Queue to do our search instead of a Stack?

Breadth-First Search

Breadth-First Search

Primary Goals

- Visit every vertex in graph $G = (V,E)$
- Construct a spanning tree for every connected component
  - **Side Effect:** Compute connected components
  - **Side Effect:** Compute a path between all connected vertices
  - **Side Effect:** Determine if the graph is connected
  - **Side Effect:** Identify cycles
- Complete in time $O(|V| + |E|)$, with memory overhead $O(|V|)$

Breadth-First Search

Primary Goals

- Visit every vertex in graph ***G* = (*V*,*E*)** **in increasing order of distance from the start**
- Construct a spanning tree for every connected component
  - **Side Effect:** Compute connected components
  - **Side Effect:** Compute a path between all connected vertices
  - **Side Effect:** Determine if the graph is connected
  - **Side Effect:** Identify cycles
  - **Side Effect: Identify shortest paths to the starting vertex**
- Complete in time $O(|V| + |E|)$, with memory overhead $O(|V|)$

BFS

```
public void BFS(Graph graph) {
  for (Vertex v : graph.vertices) {
    v.setLabel(UNEXPLORED);
  }
  for (Edge e : graph.edges) {
    e.setLabel(UNEXPLORED);
  }
  for (Vertex v : graph.vertices) {
    if (v.label == UNEXPLORED) {
      BFSOne(graph, v);
    }
  }
}
```

Same as DFS driver function…just make sure that we explore EVERY vertex, even if the graph is disconnected

```java
public void BFSOne(Graph graph, Vertex v) {
  Queue<Vertex> todo = new Queue<>();
  v.setLabel(VISITED);
  todo.enqueue(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.dequeue();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          w.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.enqueue(w);
        } else {
          e.setLabel(CROSS);
        }
}}}}
```

```java
public void BFSOne(Graph graph, Vertex v) {
  Queue<Vertex> todo = new Queue<>();
  v.setLabel(VISITED);
  todo.enqueue(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.dequeue();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          w.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.enqueue(w);
        } else {
          e.setLabel(CROSS);
        }
}}}}
```

Use a queue to keep track of what vertices we want to visit (basically a running TODO list)

```
public void BFSOne(Graph graph, Vertex v) {
  Queue<Vertex> todo = new Queue<>();
  v.setLabel(VISITED);
  todo.enqueue(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.dequeue();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          w.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.enqueue(w);
        } else {
          e.setLabel(CROSS);
        }
}}}}
```

Dequeue a vertex from the Queue and check all of it's outgoing edges

```java
public void BFSOne(Graph graph, Vertex v) {
  Queue<Vertex> todo = new Queue<>();
  v.setLabel(VISITED);
  todo.enqueue(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.dequeue();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          w.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.enqueue(w);
        } else {
          e.setLabel(CROSS);
        }
}}}}
```

When we find a new vertex, mark it as VISITED, and add it to our TODO list.

Remember, our TODO list is a Queue (FIFO) so whatever we enqueud first will be the next thing we dequeue (and explore)

```java
public void BFSOne(Graph graph, Vertex v) {
  Queue<Vertex> todo = new Queue<>();
  v.setLabel(VISITED);
  todo.enqueue(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.dequeue();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          w.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.enqueue(w);
        } else {
          e.setLabel(CROSS);
        }
}}}}
```

When doing BFS we label edges that return to visited vertices as CROSS edges

Detailed Example

UNEXPLORED

VISITED

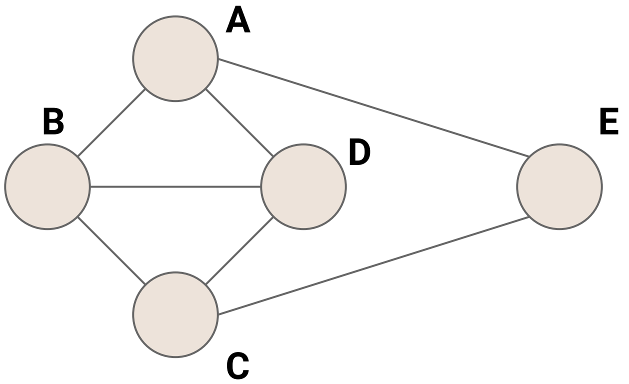
UNEXPLORED

SPANNING

CROSS

Call Stack

Work Queue



Detailed Example

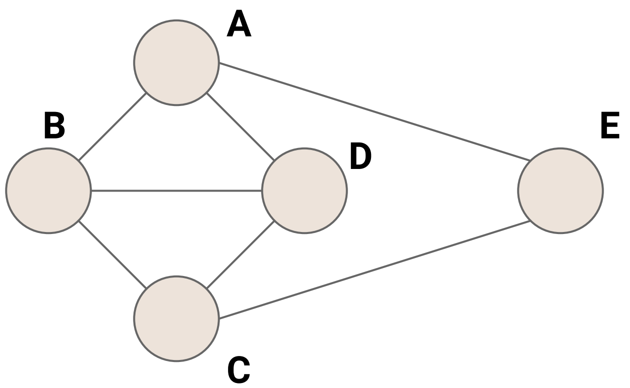
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS

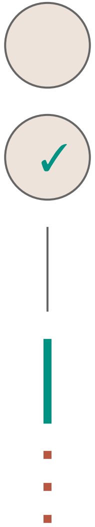


<u>Call Stack</u>

BFS(G)

<u>Work Queue</u>

Detailed Example



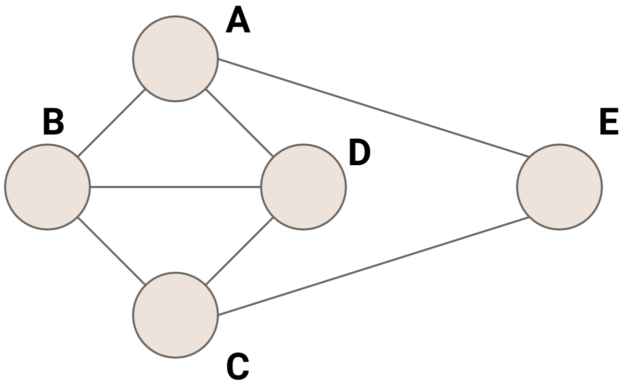
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



Call Stack

BFS(G)
BFSOne(G,A)

Work Queue

Detailed Example

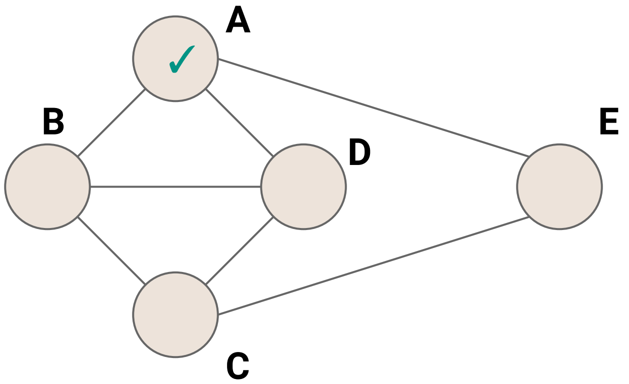
UNEXPLORED

VISITED

UNEXPLORED

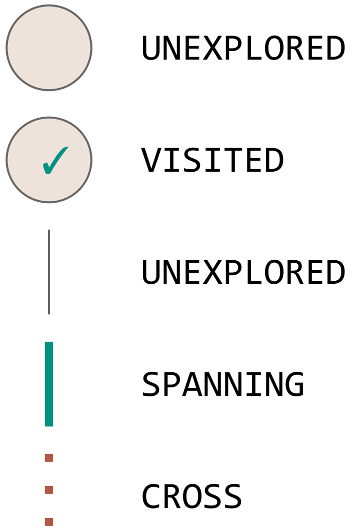
SPANNING

CROSS



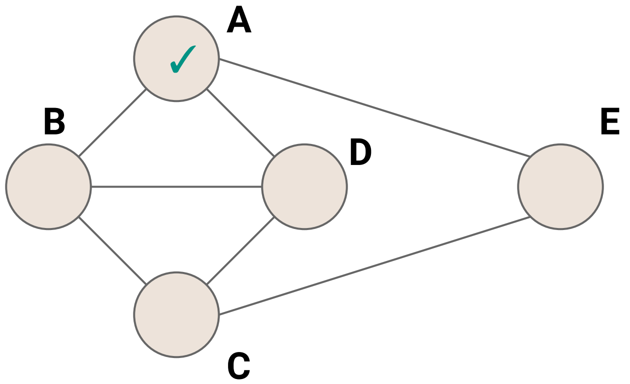
<u>Call Stack</u>	<u>Work Queue</u>
BFS(G)	**A**
BFSOne(G,A)	

Detailed Example

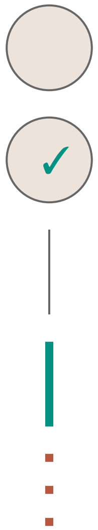


Call Stack
BFS(G)
BFSOne(G,A)

Work Queue
~~A~~



Detailed Example



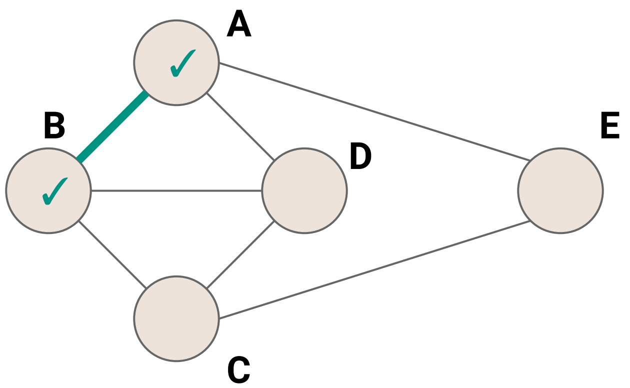
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



Call Stack

BFS(G)

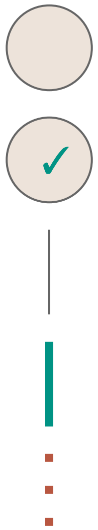
BFSOne(G,A)

Work Queue

~~A~~

B

Detailed Example



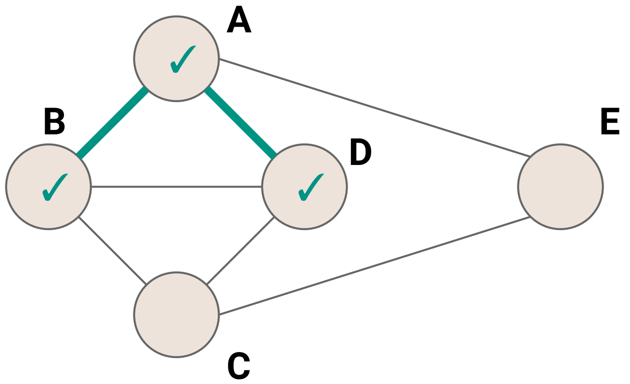
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



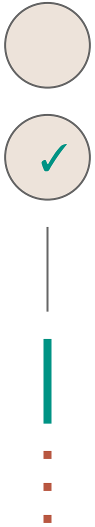
<u>Call Stack</u>

BFS(G)
BFSOne(G,A)

<u>Work Queue</u>

~~A~~
B
D

Detailed Example



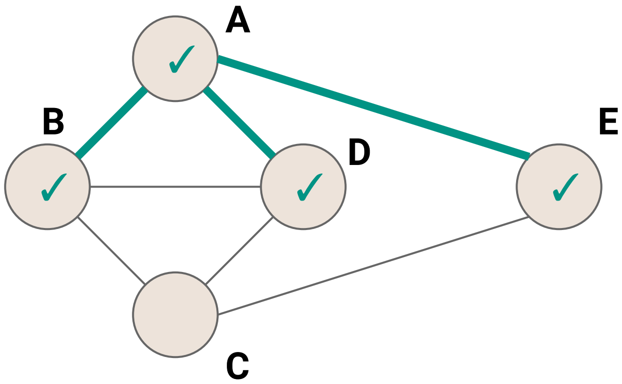
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



Call Stack

BFS(G)

BFSOne(G,A)

Work Queue

→ ~~A~~

B

D

E

Detailed Example

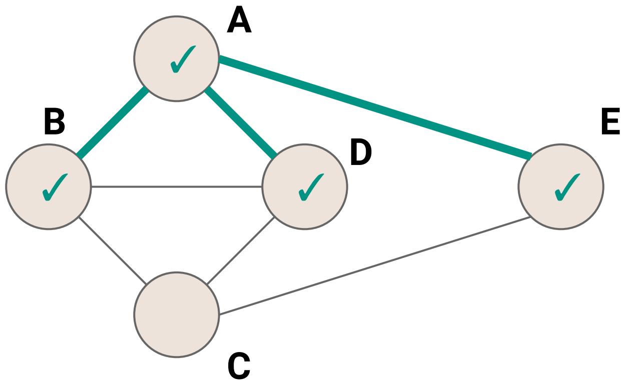
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



Call Stack

BFS(G)

BFSOne(G,A)

Work Queue

~~A~~

→ ~~B~~

D

E

Detailed Example

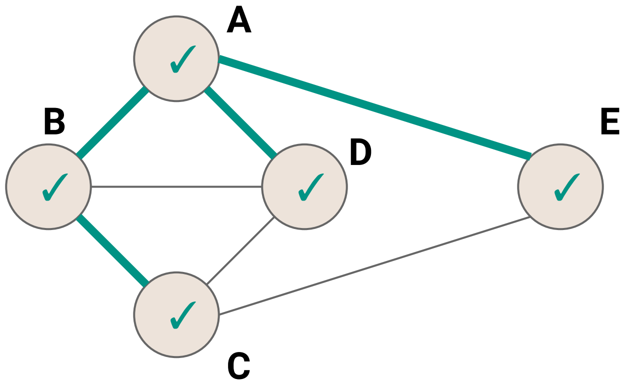
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



Call Stack
BFS(G)
BFSOne(G,A)

Work Queue
~~A~~
→ ~~B~~
D
E
C

Detailed Example

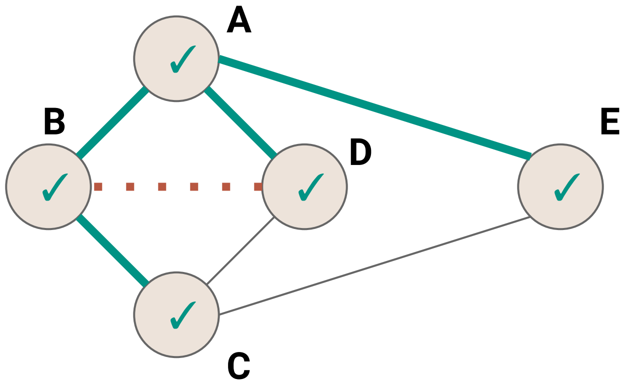
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



<u>Call Stack</u>

BFS(G)
BFSOne(G,A)

<u>Work Queue</u>

~~A~~
→ ~~B~~
D
E
C

Detailed Example

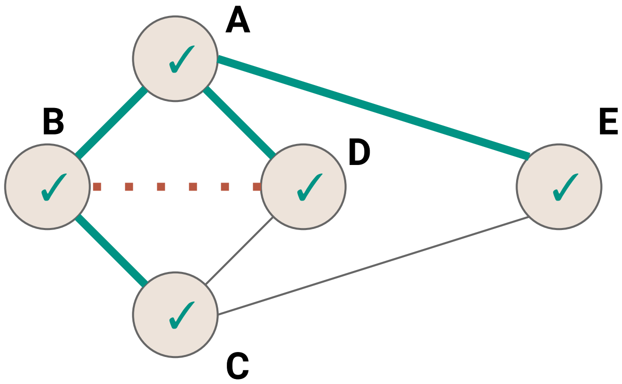
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



Call Stack

```
BFS(G)
BFSOne(G,A)
```

Work Queue

~~A~~
~~B~~
→ ~~D~~
E
C

Detailed Example

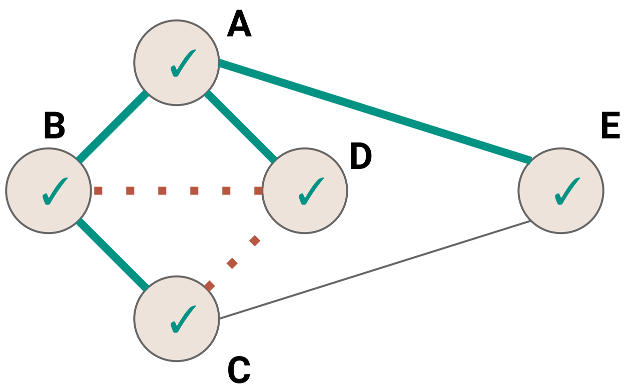
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



Call Stack

BFS(G)
BFSOne(G,A)

Work Queue

~~A~~
~~B~~
→ ~~D~~
E
C

Detailed Example

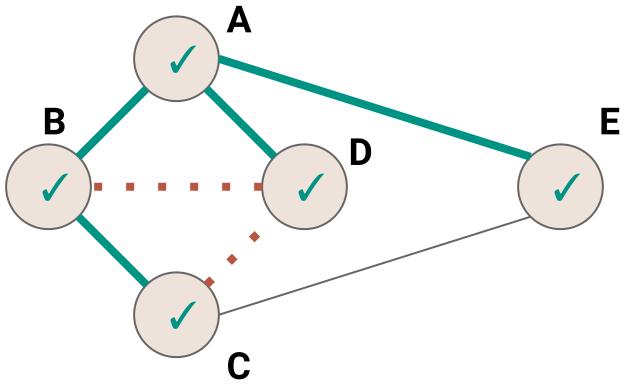
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



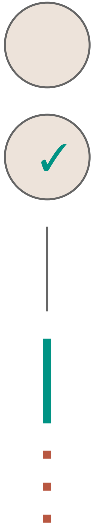
Call Stack

BFS(G)
BFSOne(G,A)

Work Queue

~~A~~
~~B~~
~~D~~
→ ~~E~~
C

Detailed Example



UNEXPLORED

VISITED

UNEXPLORED

SPANNING

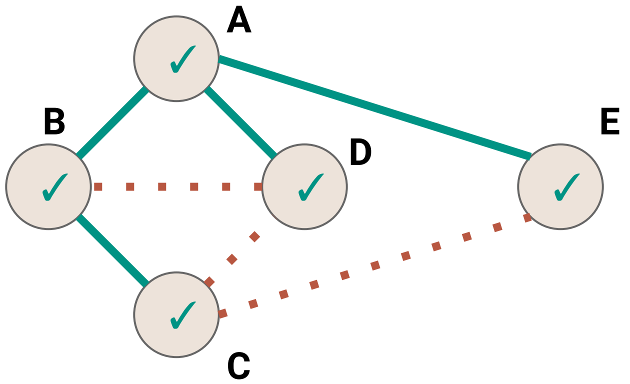
CROSS

<u>Call Stack</u>

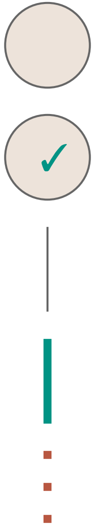
BFS(G)
BFSOne(G,A)

<u>Work Queue</u>

~~A~~
~~B~~
~~D~~
→ ~~E~~
C



Detailed Example



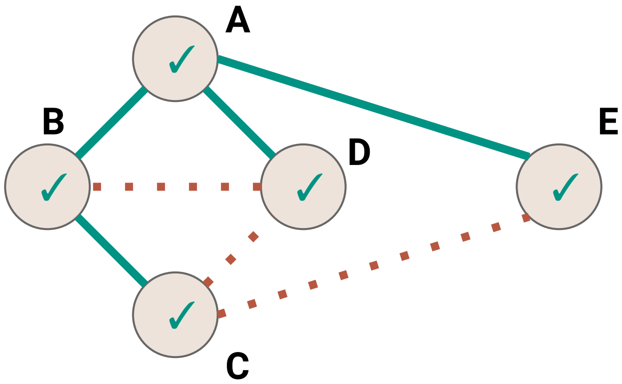
UNEXPLORED

VISITED

UNEXPLORED

SPANNING

CROSS



Call Stack

BFS(G)
BFSOne(G,A)

Work Queue

~~A~~
~~B~~
~~D~~
~~E~~
~~C~~

BFS Complexity

```
public void BFS(Graph graph) {
  for (Vertex v : graph.vertices) {
    v.setLabel(UNEXPLORED);
  }
  for (Edge e : graph.edges) {
    e.setLabel(UNEXPLORED);
  }
  for (Vertex v : graph.vertices) {
    if (v.label == UNEXPLORED) {
      BFSOne(graph, v);
    }
  }
}
```

BFS Complexity

```
public void BFS(Graph graph) {
  Θ(|V|)
  Θ(|E|)
  for (Vertex v : graph.vertices) {
    if (v.label == UNEXPLORED) {
      BFSOne(graph, v);
    }
  }
}
```

BFS Complexity

```
public void BFS(Graph graph) {
  Θ(|V|)
  Θ(|E|)
  for (Vertex v : graph.vertices) {
    if (v.label == UNEXPLORED) {
      Θ(???)
    }
  }
}
```

```java
public void BFSOne(Graph graph, Vertex v) {
  Queue<Vertex> todo = new Queue<>();
  v.setLabel(VISITED);
  todo.enqueue(v);
  while (!todo.isEmpty()) {
    Vertex curr = todo.dequeue();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          curr.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.enqueue(w);
        } else {
          e.setLabel(CROSS);
        }
}}}}
```

```
public void BFSOne(Graph graph, Vertex v) {
  Θ(1)
  while (!todo.isEmpty()) {
    Vertex curr = todo.dequeue();
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          curr.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.enqueue(w);
        } else {
          e.setLabel(CROSS);
        }
}}}}
```

```java
public void BFSOne(Graph graph, Vertex v) {
  Θ(1)
  while (!todo.isEmpty()) {
    Θ(1)
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Vertex w = e.to;
        if (w.label == UNEXPLORED) {
          curr.setLabel(VISITED);
          e.setLabel(SPANNING);
          todo.enqueue(w);
        } else {
          e.setLabel(CROSS);
        }
}}}}
```

```
public void BFSOne(Graph graph, Vertex v) {
  Θ(1)
  while (!todo.isEmpty()) {
    Θ(1)
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Θ(1)
        if (w.label == UNEXPLORED) {
          Θ(1)
          todo.enqueue(w);
        } else {
          Θ(1)
        }
}}}}
```

```
public void BFSOne(Graph graph, Vertex v) {
  Θ(1)
  while (!todo.isEmpty()) {
    Θ(1)
    for (Edge e : curr.outEdges) {
      if (e.label == UNEXPLORED) {
        Θ(1)
        if (w.label == UNEXPLORED) {
          Θ(1)
          Θ(1)
        } else {
          Θ(1)
        }
}}}}
```

```
public void BFSOne(Graph graph, Vertex v) {
  Θ(1)
  while (!todo.isEmpty()) {
    Θ(1)
    for (Edge e : curr.outEdges) {
      Θ(1)
    }
  }
}
```

```
public void BFSOne(Graph graph, Vertex v) {
  Θ(1)
  while (!todo.isEmpty()) {
    Θ(1)
    Θ(deg(v))
  }
}
```

```
public void BFSOne(Graph graph, Vertex v) {
  Θ(1)
  while (!todo.isEmpty()) {
    Θ(1)
    Θ(deg(v))
  }
}
```

How many iterations will this while loop run?

```
public void BFSOne(Graph graph, Vertex v) {
  Θ(1)
  while (!todo.isEmpty()) {
    Θ(1)
    Θ(deg(v))
  }
}
```

How many iterations will this while loop run?
Each vertex will be enqueued exactly ONCE

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  Θ(1)
  while (!todo.isEmpty()) {
    Θ(1)
    Θ(deg(v))
  }
}
```

How many iterations will this while loop run?
Each vertex will be enqueued exactly ONCE
The cost to process each vertex is deg(v)

Breadth-First Search Complexity

What is the sum over all iterations in `BFSOne`?

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$$= O(|E|)$$

Breadth-First Search Complexity

In summary…

1. Mark the vertices `UNVISITED`

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2. Mark the edges **`UNVISITED`**

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Breadth-First Search Complexity

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1. Mark the vertices **`UNVISITED`** $O(|V|)$
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3. Add each vertex to the work queue

Breadth-First Search Complexity

In summary…

1.	Mark the vertices `UNVISITED`	$O(\|V\|)$
2.	Mark the edges `UNVISITED`	$O(\|E\|)$
3.	Add each vertex to the work queue	$O(\|V\|)$

Breadth-First Search Complexity

In summary…

1.	Mark the vertices **`UNVISITED`**	$O(\|V\|)$
2.	Mark the edges **`UNVISITED`**	$O(\|E\|)$
3.	Add each vertex to the work queue	$O(\|V\|)$
4.	Process each vertex	

Breadth-First Search Complexity

In summary…

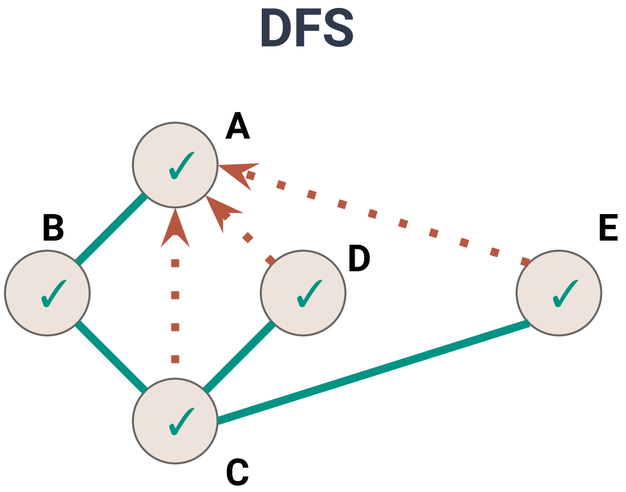
1.	Mark the vertices **`UNVISITED`**	$\boldsymbol{O(|V|)}$
2.	Mark the edges **`UNVISITED`**	$\boldsymbol{O(|E|)}$
3.	Add each vertex to the work queue	$\boldsymbol{O(|V|)}$
4.	Process each vertex	$\boldsymbol{O(|E|)}$ **total**

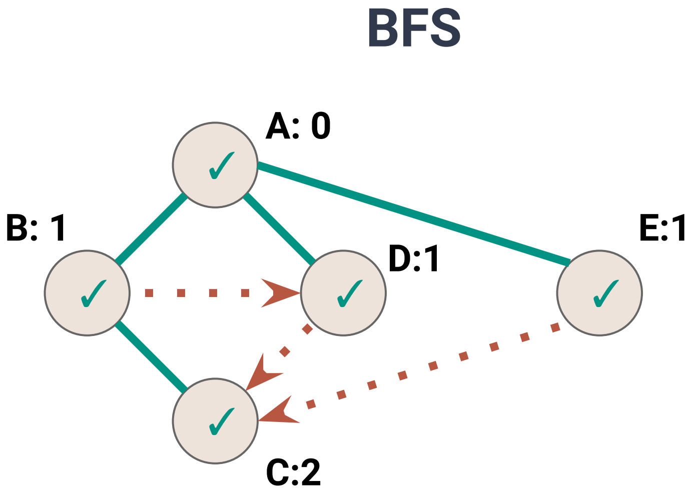
Breadth-First Search Complexity

In summary…

1.	Mark the vertices **`UNVISITED`**	***O***(|***V***|)
2.	Mark the edges **`UNVISITED`**	***O***(|***E***|)
3.	Add each vertex to the work queue	***O***(|***V***|)
4.	Process each vertex	***O***(|***E***|) **total**
		O(|***V***| + |***E***|)

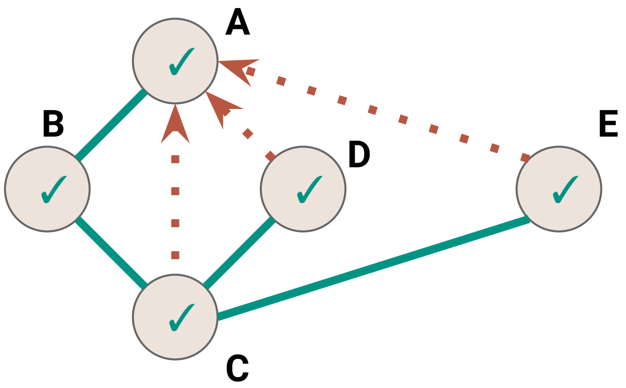
DFS vs BFS



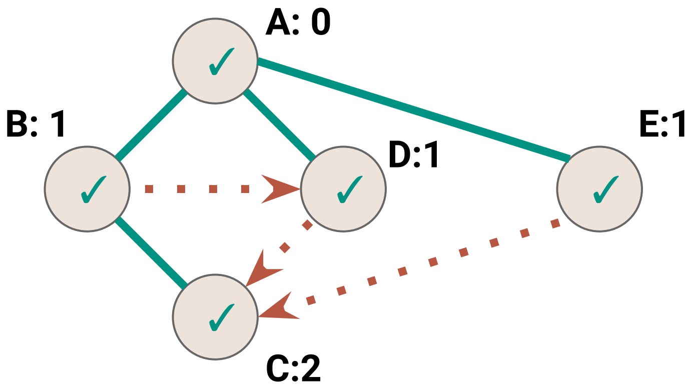


DFS vs BFS

DFS



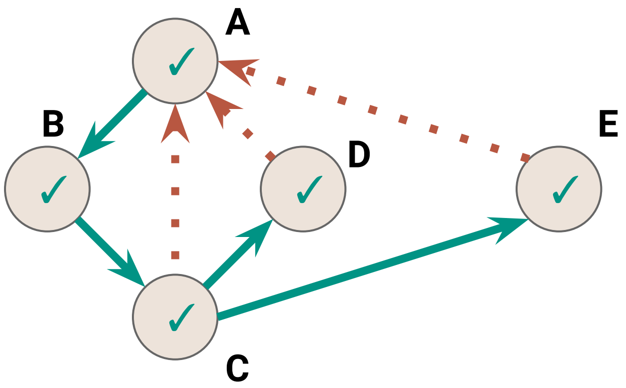
BFS



BACK Edge(*v*,*w*): ***w*** is an ancestor of ***v*** in the discovery tree

DFS vs BFS

DFS



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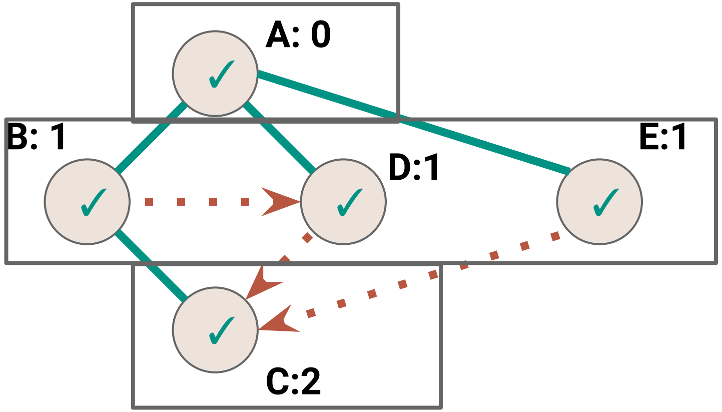
BFS



CROSS Edge(*v*,*w*): ***w*** is at the same or next level as ***v***

DFS Traversal vs BFS Traversal

Application	DFS	BFS
Spanning Trees	✓	✓
Connected Components	✓	✓
Paths/Connectivity	✓	✓
Cycles	✓	✓
Shortest Paths*		✓
Articulation Points	✓	

** we'll come back to this…*