<u>THM 1</u>: The GS algo has $\le n^2$ iterations.

① $1 \le P(t) \le U\ (<\infty)$ iteration t

② $P(t+1) \geq P(t)+1$

<u>Take 1</u>: $P(t) = $ (# ~~free~~ engaged ~~men~~ men & women) $\cancel{+1}$.

$2 \le P(t) \le 2n\ \cancel{+1}\ \forall t \geq 1$ ① ✓

Problem there can be t s.t. $P(t+1) = P(t)$.

② ✗

<u>Take 2</u>: $P(t) = t$

$P(t+1) = \cancel{P(t)}\,t + 1 = P(t)+1$ ② ✓

Problem: ① ✗

<u>Take 3</u>: $P(t) = $ # proposals.

②: In every iteration, a proposal (w, m) is made + it is a new proposal

$\Rightarrow P(t+1) = P(t)+1$

① $P(t) \geq 1$ for every $t \geq 1$

$P(t) \leq$ # pairs (w, m) s.t $w \in W, m \in M$
$= n^2$

THM 2: The GS algo always outputs a stable matching. (Let S be the output)

Pf. idea: We will prove

Lemma 1: S is a perfect matching

Lemma 2: There is no instability in S.

Proof idea (Lemma 1):

Obs 3: S is a matching.

Proof by contradiction: Assume S is not a perfect matching $\Rightarrow$ the algorithm has not terminated.

~~Proposition~~ Lemma 3: If S is not a perfect matching $\Rightarrow$ $\exists$ a free woman w, who has not proposed to all men.

Pf idea (Lemma 3): Proof by contradiction + Obs 1 + pigeon-hole principle

Pigeon-hole principle: If there are x holes & $\geq x+1$ pigeons $\Rightarrow \exists$ ~~the~~ hole w/ ≥ 2 pigeons.

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— $\leq x-1$ pigeon $\Rightarrow$ __________ 0 pigeons.

Pf details (Lemma 3): (Given the S is not a perfect matching)

$\Rightarrow \exists$ a free woman w proposed to all men.

By contradiction

$\exists$ free woman w who has proposed to all men.

(1) ~~When~~ After w proposed to m, m was engaged ($\forall m \in M$)

2) By Obs(1), all n men are engaged.

(3) But $\leq n-1$ women are engaged

by P.H.P $\Rightarrow \exists$ m who is not engaged

Contradict(2)