RECAP

GAMES: (i) n players (ii) Strategy sets: $A_1, \ldots, A_n$ (iii) Utility functions: $\forall i, u_i : \prod_j A_j \to \mathbb{R}$

MIXED STRATEGIES: $\Delta(A_i)$: set of all prob. dist. over $A_i$

$u_i(p_1, \ldots, p_n) = \sum_{(a_1, \ldots, a_n) \in \prod_j A_j} u_i(a_1, \ldots, a_n) \cdot p_1(a_1) \cdot p_2(a_2) \cdots p_n(a_n)$

NE: $(p_1, \ldots, p_n) \in \prod_j \Delta(A_j)$ is a NE if $\forall i$ & $\forall p_i' \in \Delta(A_i)$

$u_i(p_i', k_i) \leq u_i(k_i, k_i)$

Criticism of NE

(i) If multiple NEs, which one will occur?

(ii) May not even be able to state the stable pay off.

(iii) Sort of circular definition
2 player 0-sum game

(i) \( n = 2 \)  (ii) \( A_1, A_2 \)  (iii) \( u_2(q_1, q_2) = -u_1(q_1, q_2) \) for \( (q_1, q_2) \in A_1 \times A_2 \)

Given \( q \in \Delta(A_2) \), player 1 would like to
\[
\arg \max_{p \in \Delta(A_1)} u_1(p, q)
\]

Given \( p \in \Delta(A_1) \), player 2 would like to
\[
\arg \min_{q \in \Delta(A_2)} u_1(p, q)
\]
von Neumann's Minimax Thm:

\( \forall \) 2 player 0-sum games with \( u_1 : A_1 \times A_2 \rightarrow \mathbb{R} \),
\( \exists \) an \( v \in \mathbb{R} \) (value of the game) s.t.

(i) \( v = \max_{p \in \Delta(A_1)} \min_{q \in \Delta(A_2)} u_1(p, q) = \min_{q \in \Delta(A_2)} \max_{p \in \Delta(A_1)} u_1(p, q) \)

(ii) \( \exists \) a mixed NE. \( \forall \) NE \( (p, q) \in \Delta(A_1) \times \Delta(A_2) \)
\( p \in \arg \max \min u_1(p', q) \) iff
\( q \in \arg \min \max u_1(p, q') \)

(iii) \( \forall \) NE \( (p, q), v = u_1(p, q) \)
Main Lemma: (...) \[ \max_{p'} \min_{q'} u_1(p', q') = L \geq \min_{q} \max_{p} u_1(p, q) = R \]

Pf of (i) of Thm:

(\geq): Main Lemma

(\leq): for any \((\hat{p}, \hat{q})\), \[ u_1(\hat{p}, \hat{q}) \leq \max_{p} u_1(p, q) \]

\[ q^* = \arg \min_{q'} \max_{p'} u_1(p', q') \]

\[ \min_{q'} u_1(\hat{p}, q') \leq u_1(\hat{p}, q^*) \leq \max_{p'} u_1(p', q^*) = R \]

\[ \text{As true for } \hat{p}, \]

\[ L \leq R \]
Pf of (ii) & (iii)

\[ B_1 \overset{\text{def}}{=} \arg \max \min_{p', q'} u_1(p', q') \]
\[ B_2 \overset{\text{def}}{=} \arg \min \max_{q', p'} u_1(p', q') \]

Claim: \( B_1, B_2 \neq \emptyset \)

\( B_1 \neq \emptyset \iff \text{for every fixed } q, \exists a p \text{ that } \max_{p'} u_1(p, q) \)

\( \sim B_2 \neq \emptyset \) (Formal: compactness of \( \Delta(A_1), \Delta(A_2) \), \( A_1 \& A_2 \) are finite \& \( u_1 \) is continuous.)
\[(p, q) \in B_1 \times B_2 \implies (p, q) \text{ is NE} \checkmark \]

1. player 1 plays \( p' \neq p \)
   \[ u_1(p', q) \leq u_1(p, q) \quad (\text{as } p \in B_1) \]
   \( \sim \) player 2 plays \( q' \neq q \), \[ u_1(p, q') \geq u_1(p, q) \]

2. \[ v = \max_{p'} \min_{q'} u_1(p', q') = \min_{q'} \max_{p'} u_1(p', q') \quad \uparrow \quad q' \]
   \[ \leq u_1(p, q) \leq \max_{q} \min_{p} u_1(p, q') \quad \checkmark \]
   \( \leq u_1(p, q) \leq \max_{q} \min_{p} u_1(p, q) = \min_{q} \max_{p} u_1(p, q) \quad \hat{p} \]
   \[ \hat{p} \in B_1, \quad \hat{q} \in B_2 = v \]

(i)
\[(p, q) \text{ NE } \Rightarrow (p, q) \in R_1 \times B_2\]
\[\& \quad u_1(p, q) = v \quad q \in B_2\]
\[u_1(p, q) = \max_{p'} u_1(p', q) \quad \Rightarrow \min_{q'} \max_{p'} u_1(p', q')\]
\[u_1(p, q) = \min_{q} u_1(p, q') \quad \max_{p} \min_{q} u_1(p, q)\]
\[\Rightarrow u_1(p, q) = v\]
\[\therefore p \in B_1\]
Proof of Main Lemma (2 proofs)

\[ T = \lceil 4 \frac{\ln n}{\delta^2} \rceil \]

Regret of (Max)Hedge (3 = \sqrt{\frac{\ln n}{T}})

\[ \mathbb{E} \left[ \sum_{t=1}^{T} g_t(x_t) \right] \geq \mathbb{E} \left[ \max_{x \in [n]} \sum_{t=1}^{T} g_t(x) \right] \]

\[ -2 \sqrt{T \ln n} \]

\[ T \delta \mathbb{E} \left[ \max_{x \in \Delta([n])} \sum_{t=1}^{T} g_t(x) \right] \]
\[
\max_{p'} \min_{q'} u_1(q', q'') \geq \min_{\hat{q}} \max_{\hat{p}} u_1(\hat{p}, \hat{q})
\]

W.l.o.g. \( u_1(a_1, a_2) \in [0, 1] \) \( \forall (a_1, a_2) \in A_1 \times A_2 \)

\[
(\text{pick } b \& c \text{ s.t. } n = \max \{ |A_1|, |A_2| \} \quad u'_1 = b u_1 + c)
\]

Player 1 runs \( \text{Hedge}(\varepsilon) \)

each \( q_i \in A_1 \) is an expert

\( P_1, \ldots, P_T \)

\( g_t(x) = u_1(x, q_t) \)

Player 2 runs \( \text{Hedge}(\varepsilon) \)

each \( q_2 \in A_2 \) is an expert

\( q_1, \ldots, q_T \)

\( g_t(x) = 1 - u_1(P_t, x) \)
Player 1

By Hedge's regret thm:

\[ \frac{1}{T} \sum_{t=1}^{T} u_1 (P_t, q_t) \geq \max_P \frac{1}{T} \sum_{t=1}^{T} u_1 (P_t, q_t) - \delta \]

\[ P = \frac{1}{T} \sum_{t=1}^{T} P_t \]

for any fixed \( q \)

\[ \frac{1}{T} \sum_{t=1}^{T} u_1 (P_t, q_t) = u_1 (\bar{P}, q) \]

Player 2

\[ \frac{1}{T} \sum_{t=1}^{T} -u_1 (P_t, q_t) \]

\[ \geq \max_q \frac{1}{T} \sum_{t=1}^{T} -u_1 (P_t, q_t) - \delta \]

\[ q = \frac{1}{T} \sum_{t=1}^{T} q_t \]

for any fixed \( P \)

\[ \frac{1}{T} \sum_{t=1}^{T} u_1 (P_t, q_t) \leq u_1 (\bar{P}, \bar{q}) \]

\[ \bar{q} = \frac{1}{T} \leq q_t \]
\[
\min_{p, q} \frac{1}{T} \sum_{t=1}^{T} u_1(P_t, q_t) + \delta \geq \frac{1}{T} \sum_{t=1}^{T} u_1(P_t, q_t) \\
\geq \max_{p} \frac{1}{T} \sum_{t=1}^{T} u_1(P, q_t) - \delta \\
\geq \frac{1}{T} \sum_{t=1}^{T} u_1(\hat{p}, \hat{q}_t) - \delta \\
\Rightarrow \min_{q} u_1(\hat{p}, \hat{q}) + \delta \geq \max_{p} u_1(p, \bar{q}) - \delta \\
\max_{p} \min_{q} u_1(P, q) \geq \min_{q} \max_{p} u_1(p, q)
\]
\[
\max_p \min_q u_1(p,q) + \delta \\
\geq \frac{1}{T} \sum_{t=1}^{T} u_1(p_t, q_t) \\
\geq \min_q \max_p u_1(p,q) - \delta \\
\geq \min_q \max_p u_1(p,q) - \delta \\
T = \Theta \left( \frac{\ln n}{\delta^2} \right)
\]
Yao's lemma (prove lower bounds for randomized algo).

\[ I \rightarrow \text{set of all inputs } \mathcal{A} = \Delta(I) \]

\[ \mathcal{A} \rightarrow \text{deterministic algorithms.} \]

ie \( I, a \in A \Rightarrow t(i,a) \rightarrow \text{cost of running } a \text{ on } i. \)

\[ R = \Delta(A) = \text{set of all randomized algos.} \]

\[ \min_{r \in R} \max_{i \in I} t(i,r) = \min_{r \in R} \max_{a \in A} t(d,r) \]

\[ = \max_{d \in \text{deg}} \min_{r \in R} t(d,r) = \max_{a \in A} \min_{d \in \text{deg}} t(d,a) \]

\[ \uparrow \text{deg } r \in R \]

\[ \downarrow \text{pick } d \in \text{deg } \]

\[ \min_{a \in A} t(d,a) \]