Is BQP Squeezed Out or In?

UB CSE Theory Group

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6 May, 2024

[1]Includes joint work with Amlan Chakrabarti, U. Calcutta, and Chaowen Guan, U. Cincinnati

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5. There is a tight deterministic time hierarchy...but the languages involved are diagonal or are "artificially complete."

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- *Proved* when G is a finitely-generated subgroup of a connected Lie group. (J. Tits).

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- More-natural characterizations, or indelibly "Meta"?

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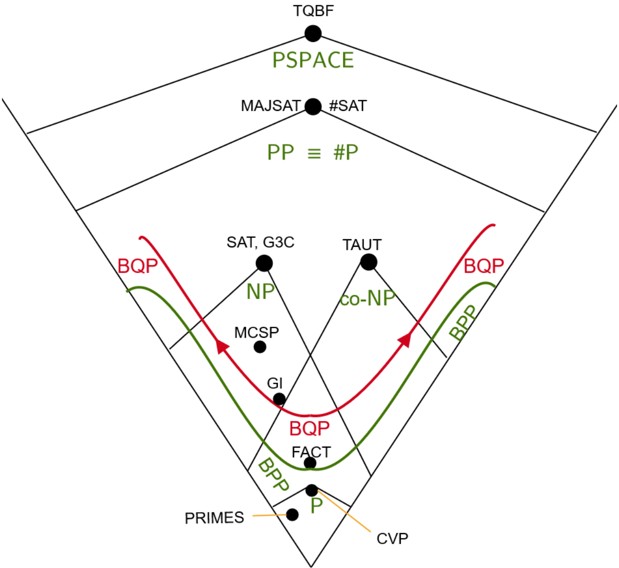
- #P is polynomial-time *Turing*-equivalent to the language class **PP**, which is characterized by languages of the form $L_h = \{ (x, k) : h(x) \geq k \}$ over $h \in$ #P.
- PP is the lowest known "simple" upper bound for **BQP**, bounded-error quantum polynomial time. (A technical subclass called **AWPP** contains BQP.)

Diagram of These Classes and Problems



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$$Z_A(G) = \sum_{h:V \to [m]} \prod_{(u,v) \in E} A[h(u), h(v)]$$

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  where $G = (V,E)$ on n nodes and A is a symmetric $m \times m$ matrix.
- For many other counting problems, seemingly small changes in settings flip the problem between P and #P-hard, with no sign of anything in between.

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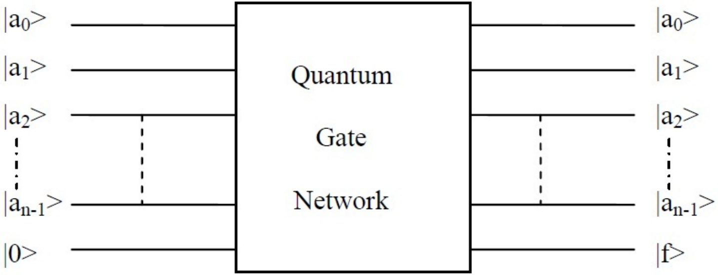
This brings up our main philosophical question:

> If there is "nothing natural" between P and #P-complete, where does that leave BQP?

(For this purpose, NP is tantamount to #P.)

Quantum Circuits

Quantum circuits look more constrained than Boolean circuits:

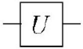


But Boolean circuits look similar if we do Savage's TM-to-circuit simulation and call each *column* for each tape cell a "cue-bit."

Quantum Gates—three slides by M. Rötteler

Quantum gates

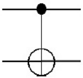
single qubit operation:



controlled-NOT:

control

target

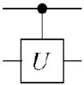


unitary matrix $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

controlled-U:

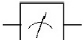
control

target



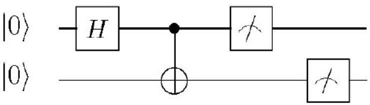
unitary matrix $= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{pmatrix}$

measurement in the $|0\rangle, |1\rangle$ basis:



Quantum circuit example

$$H \otimes \mathbf{1}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \mathbf{1}_2$$



$\text{Prob}(\text{meas. } |00\rangle) = 0.5$

$\text{Prob}(\text{meas. } |11\rangle) = 0.5$

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle$$

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$$|00\rangle \to |00\rangle$$
$$|01\rangle \to |01\rangle$$
$$|10\rangle \to |11\rangle$$
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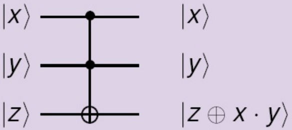
Toffoli Gate

The Toffoli gate "TOF"

x	y	z	x'	y'	z'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0



Theorem (Toffoli, 1981)

Any reversible computation can be realized by using TOF gates and ancilla (auxiliary) bits which are initialized to 0.

Slides by Martin Rötteler

Some More Gates

$$\mathsf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathsf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \mathsf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathsf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \mathsf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}, \quad \mathsf{R}_8 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/8} \end{bmatrix},$$

$$\mathsf{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathsf{CZ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathsf{CS} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}.$$

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- The gates $\mathsf{H}, \mathsf{X}, \mathsf{Y}, \mathsf{Z}, \mathsf{S}, \mathsf{CNOT}, \mathsf{CZ}$ generate *Clifford circuits*, which are simulable in polynomial time.

Some More Gates

$$\mathsf{X} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathsf{Y} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \mathsf{Z} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\mathsf{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \mathsf{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}, \quad \mathsf{R}_8 = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/8} \end{bmatrix},$$

$$\mathsf{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathsf{CZ} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathsf{CS} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}.$$

- The gates $\mathsf{H}, \mathsf{X}, \mathsf{Y}, \mathsf{Z}, \mathsf{S}, \mathsf{CNOT}, \mathsf{CZ}$ generate *Clifford circuits*, which are simulable in polynomial time.
- Adding any of T, R_8, or CS gives the full power of BQP.

Bounded-error Quantum Poly-Time

A language A belongs to BQP if there are uniform poly-size quantum circuits C_n with n data qubits, plus some number $\alpha \geq 1$ of "ancilla qubits," such that for all n and $x \in \{0,1\}^n$,

$$\begin{aligned} x \in A &\implies \Pr[C_n \text{ given } \langle x0^\alpha| \text{ measures 1 on line } n+1] > 2/3; \\ x \notin A &\implies \Pr[\ldots] < 1/3. \end{aligned}$$

One can pretend $\alpha = 0$ and/or measure line 1 instead. One can also represent the output as the "triple product" $\langle b \mid C \mid a \rangle$, with $a = x0^\alpha$, $b = 0^{n+\alpha}$.

Two major theorems about BQP are:

(a) C_n can be composed of just Hadamard and Toffoli gates [Y. Shi].

(b) Factoring is in BQP [P. Shor].

More-general forms of a known relation

- Assume all nonzero entries $re^{i\theta}$ of gate matrices in quantum circuits C have equal magnitude $|r|$ and θ an integer multiple of $2\pi/K$.
- Suppose C has h Hadamard gates as nondeterministic games.
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Theorem (multiplicative form, case $G = \mathbb{F}_2$ is Dawson et al. (2004) + ...)

Any QC C of n qubits can be quickly transformed into a polynomial P_C of the form $\prod_g P_g$ and a constant $R > 0$ such that for all $x, z \in \{0,1\}^n$:

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Here g ranges over all gates and outputs of C and y ranges over $\{0,1\}^h$.

Degree is $\Theta(s)$ where s is the number of gates in C.

Additive Case

Theorem (RCG (2017), RC (2007-9), cf. Bacon-van Dam-Russell (2008))

Given C and K, we can efficiently compute a polynomial $Q_C(x_1, \ldots, x_n, y_1, \ldots, y_h, z_1, \ldots, z_n, w_1, \ldots, w_t)$ of degree $O(1)$ over $\mathbb{Z}_K$ and a constant R' such that for all $x, z \in \{0,1\}^n$:

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- Hence poly-time simulation by solution counting in $\mathbb{Z}_4$.

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- IQP circuits use Hadamard gates only at the beginning and end of the circuit, CS gates, and diagonal one-qubit gates.
- The same idea with only CZ and (optionally, for self-loops) Z gates between the two banks of Hadamards are called **graph-state circuits**, and are equivalent to general Clifford circuits in power.

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- General observations—how wide are the possibilities and prospects?

-

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- Logic-based full QC simulator, 8,000+ lines of C++. [show demo]

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- Also give $O(N)$-time reduction ($N = n^2$) from computing $n \times n$ matrix rank over $\mathbb{F}_2$ to the QC simulation.

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- Classical simulation of n qubits takes $O(n^2)$ time per single-qubit measurement [Aaronson-Gottesman, 2004], $O(n^3)$ time to measure all n qubits.
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- Also give $O(N)$-time reduction ($N = n^2$) from computing $n \times n$ matrix rank over $\mathbb{F}_2$ to the QC simulation.
- Means that the n^2-vs.-n^ω weak/strong simulation gap canot be closed unless matrix rank is in $O(n^2)$ time over $\mathbb{F}_2$.

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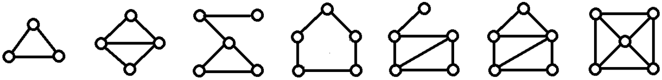
- Also yields an apparently new class of undirected graphs:



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- Second main purpose of simulator [show] is to enable tinkering with approximative methods.

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- Singular points of varieties determine (most of?) amplitude under the Principle of Least Action, conjectured by Bacon, van Dam, and Russell [2008, unpublished]).

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- Involvement in the general debate over *Quantum Advantage*.