For general homework policies and our suggestions, please see the homework policy document. You can and will lose points if you do not follow the policies.

For this homework, it might help if you read the the subsection “Extensions” in Section 1.1 of the textbook.

1. (You must work on this problem on your own: NO collaboration is allowed) (40 points)
   (a) (20pts) Decide whether the following statement is true or false:
       For any instance of the Stable Matching Problem, if there exists a perfect matching that is not stable due to \( m \) and \( w \) wanting to be together, then \( (m, w) \) will be in every stable matching of that instance. In other words, if \( (m, w') \) and \( (m', w) \) are both in a perfect matching and \( m \) prefers \( w \) over \( w' \) and \( w \) prefers \( m \) over \( m' \), then \( (m, w) \) will be in every stable matching.
       If you state true then you will have to formally argue why the statement is correct. If you state false, then you have give a counter-example.
   (b) (20pts) Exercise 1 in Chapter 1:
       Decide whether the following statement is true or false:
       In every instance of the Stable Matching Problem, there is a stable matching containing a pair \( (m, w) \) such that \( m \) is ranked first on the preference list of \( w \) and \( w \) is ranked first on the preference list of \( m \).
       If you state true then you will have to formally argue why the statement is correct. If you state false, then you have give a counter-example.

2. (45 points) Exercise 4 in Chapter 1:
   Gale and Shapley published their paper on the Stable Matching Problem in 1962; but a version of their algorithm had already been in use for ten years by the NAtional Resident Matching Program, for the problem of assigning medical residents to hospitals.
   Basically, the situation was the following. There were \( m \) hospitals, each with a certain number of available positions for hiring residents. There were \( n \) medical students graduating in a given year, each interested in joining one of the hospitals. Each hospital had a ranking of the students in order of preference, and each student had a ranking of the hospitals in order of preference. We will assume that there were more students graduating than there were slots available in the \( m \) hospitals.
   The interest, naturally, was in finding a way of assigning each student to at most one hospital, in such a way that all available positions in all hospitals were filled. (Since we are assuming a surplus of students, there would be some students who do not get assigned to any hospital.)
   We say that an assignment of students to hospitals is stable if neither of the following situations arises.
• First type of instability: There are students $s$ and $s'$, and a hospital $h$, such that
  - $s$ is assigned to $h$, and
  - $s'$ is assigned to no hospital, and
  - $h$ prefers $s'$ over $s$.

• Second type of instability: There are students $s$ and $s'$, and hospitals $h$ and $h'$, such that
  - $s$ is assigned to $h$, and
  - $s'$ is assigned to $h'$, and
  - $h$ prefers $s'$ over $s$, and
  - $s'$ prefers $h$ over $h'$.

So we basically have the Stable Matching Problem, except that (i) hospitals generally want more than one resident, and (ii) there is a surplus of medical students.

Show that there is always a stable assignment of students to hospitals, and give an algorithm to find one.

Show there is always a stable matching: 15pts
Give an algorithm to solve the problem: 15pts
Prove the algorithm is correct: 15pts

3. (15 points) In real life there are divorces and I mentioned that the stable marriage problem does not handle divorces. This is because we assume everyone is interested in everyone else of the opposite sex and we assume that the preferences do not change.

In this problem, we will see the effect of changes in preferences in the outcome of the Gale-Shapley algorithm (for this problem you can assume the version of the Gale-Shapley algorithm that we did in class where the women do all the proposing).

Given an instance of the stable marriage problem (i.e. set of men $M$ and the set of women $W$ along with their preference lists: $L_m$ and $L_w$ for every $m \in M$ and $w \in W$ respectively), call a man $m \in M$ a home-wrecker if the following property holds. There exists an $L'_m$ such that if $m$ changes his preference list to $L'_m$ (from $L_m$) then the Gale-Shapley algorithm matches everyone to someone else. In other words, let $S_{\text{orig}}$ be the stable marriage output by the Gale-Shapley algorithm for the original input and $S_{\text{new}}$ be the stable marriage output by the Gale-Shapley algorithm for the new instance of the problem where $m$’s preference list is replaced by $L'_m$ (but everyone else has the same preference list as before). Then $S_{\text{orig}} \cap S_{\text{new}} = \emptyset$.

For every integer $n \geq 2$ prove the following: There exists an instance of the stable marriage problem with $n$ men and $n$ women such that there is a man who is a home-wrecker.

(Note: To get full credit you must present an example for every $n \geq 2$, that is, you have to present a “family” of examples. Further, your proof argument should work for every value of $n \geq 2$.)

(Hint: If it helps, you can assume that when there are multiple free woman, the Gale-Shapley algorithm will pick the one that leads to your claimed $S_{\text{new}}$ and $S_{\text{old}}$. In other words, in the iterations where the Gale-Shapley algorithm has a choice you can decide which free woman it picks.)
Give an instance for any $n$: 5pts
Prove this for all $n$: 10pts