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Lecture 3: Stochastic Processes

Some texts on Probability Theory are [1, 5–8]. The absolute classic is William Feller's [2, 3]. There are also two excellent free books on the Internet at

- `http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html`,
- `http://www.mit.edu/people/dimitrib/Probability.html`

A good book on Markov Chain is [4].

Definition 1. A *stochastic process* is a collection of random variables indexed by some set T: $\{X(t), t \in T\}$. Elements of the *index set* T are often thought of as a point in time, and thus $X(t)$ is refered to as the *state* of the process at time t. The set of all posible values of the $X(t)$ are called the *state space* of the process. Naturally, when T is countable the process is said to be *discrete-time*; while if T is an interval of the real line, then the process is called a *continuous-time* process. The set of all possible values of the variables $X(t)$ is called the *state space* of the process.

Example 2 (Bernulli process). A sequence $\{X_1, X_2, \dots\}$ of independent Bernulli random variables each with parameter p is called a *Bernulli Process*. Recall that $P[X_i = 1] = p$, and $P[X_i = 0] = 1 - p$ for each i. Each X_i could be thought of as an indicator variable for a successful trial at time i or of an arrival of an item (customer) at time i. Also recall that

$$\begin{aligned} E[X_i] &= p \\ \text{Var}\,[X_i] &= p(1-p), \forall i. \end{aligned}$$

We are often interested in several stochastic processes associated with a Bernulli process: $\{S_n, n \geq 0\}$, $\{T_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$. Here, $S_n = \sum_{i=1}^n X_n$ is nothing but the the number of arrivals in n time slots, T_n the number of slots from after the $(n-1)$st arrival up to and including the nth arrival, and $Y_n = \sum_{i=1}^n T_i$ the number of slots from the beginning up to and including the nth arrival.

For each random variable X, we use $p_X(\cdot)$ to denote the probability mass function of X, namely $p_X(k) = P[X = k]$. It is then obvious that

$$\begin{aligned} p_{S_n}(k) &= \binom{n}{k} p^k (1-p)^{n-k}, 0 \leq k \leq n \\ p_{T_n}(k) &= (1-p)^{n-1} p \\ p_{Y_n}(k) &= \binom{k-1}{n-1} p^n (1-p)^{k-n}, k \geq n. \end{aligned}$$

The mass function of Y_n is called the *Pascal probability mass function* of order n. From these we can calculate the expectations and variances of the variables of interest as follows.

$$\begin{aligned} E[S_n] &= \sum_{k=0}^n k \binom{n}{k} p^k (1-p)^{n-k} \\ &= np \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} \\ &= np. \end{aligned}$$

We have used the *binomial formula*:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

We actually calculated $E[S_n]$ the hard way. Recall the *linearity of expectation*, namely

$$\text{if } X = \sum_i a_i X_i, \text{ then } E[X] = \sum_i a_i E[X_i],$$

and that $S_n = \sum_{i=1}^n X_i$. We have

$$E[S_n] = \sum_{i=1}^{n} E[X_i] = np.$$

The variance of S_n can be calculated similarly

$$\begin{aligned}
\text{Var}\,[S_n] &= E[S_n^2] - (E[S_n])^2 \\
&= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k} - (np)^2 \\
&= np \sum_{k=1}^{n} k \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} - (np)^2 \\
&= np \sum_{k=1}^{n} (k-1) \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} + np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} - (np)^2 \\
&= np \sum_{k=0}^{n-1} (k-1) \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} + np \sum_{k=0}^{n-1} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} - (np)^2 \\
&= np(n-1)p + np - (np)^2 \\
&= np(1-p).
\end{aligned}$$

T_n is just a geometric random variable, hence

$$\begin{aligned}
E[T_n] &= \frac{1}{p} \\
\text{Var}\,[T_n] &= \frac{1-p}{p^2}.
\end{aligned}$$

Lastly, $Y_n = \sum_{i=1}^n T_i$ implies

$$E[Y_n] = \frac{n}{p}.$$

The variance is slightly more difficult to compute. We need to make use of the following binomial formula:

$$\frac{1}{(1-x)^n} = (1-x)^{-n} = \sum_{i=0}^{\infty} \binom{i+n-1}{n-1} x^i. \tag{1}$$

Moreover, differentiating both sides we get

$$\sum_{i=1}^{\infty} i \binom{i+n-1}{n-1} x^{i-1} = n(1-x)^{-n-1}. \tag{2}$$

Differentiating both sides again yields

$$\sum_{i=2}^{\infty}(i^2-i)\binom{i+n-1}{n-1}x^{i-2}=n(n+1)(1-x)^{-n-2}. \tag{3}$$

Now, in the following calculation we let $i=k-n$ and $q=1-p$ from line 3 on:

$$\begin{aligned}
\text{Var}\,[Y_n] &= E[T_n^2]-(E[T_n])^2 \\
&= \sum_{k=n}^{\infty}k^2\binom{k-1}{n-1}p^n(1-p)^{k-n}-\left(\frac{n}{p}\right)^2 \\
&= p^n\sum_{i=0}^{\infty}(n+i)^2\binom{i+n-1}{n-1}q^i-\left(\frac{n}{p}\right)^2 \\
&= n^2p^n\sum_{i=0}^{\infty}\binom{i+n-1}{n-1}q^i+2np^n\sum_{i=0}^{\infty}i\binom{i+n-1}{n-1}q^i+p^n\sum_{i=0}^{\infty}i^2\binom{i+n-1}{n-1}q^i-\left(\frac{n}{p}\right)^2 \\
&= n^2p^n(1-q)^{-n}-\left(\frac{n}{p}\right)^2+2np^n\sum_{i=0}^{\infty}i\binom{i+n-1}{n-1}q^i \\
&\quad +p^n\sum_{i=0}^{\infty}i\binom{i+n-1}{n-1}q^i+p^n\sum_{i=0}^{\infty}(i^2-i)\binom{i+n-1}{n-1}q^i \\
&= n^2\left(1-\frac{1}{p^2}\right)+(2n+1)p^n\sum_{i=0}^{\infty}i\binom{i+n-1}{n-1}q^i+p^n\sum_{i=0}^{\infty}(i^2-i)\binom{i+n-1}{n-1}q^i.
\end{aligned}$$

Now, we could use formulas (1), (2), and (3) to finish the calculation:

$$\begin{aligned}
\text{Var}\,[Y_n] &= n^2\left(1-\frac{1}{p^2}\right)+(2n+1)p^nq\sum_{i=0}^{\infty}i\binom{i+n-1}{n-1}q^{i-1}+p^nq^2\sum_{i=0}^{\infty}(i^2-i)\binom{i+n-1}{n-1}q^{i-2} \\
&= n^2\left(1-\frac{1}{p^2}\right)+(2n+1)p^nqn(1-q)^{-n-1}+p^nq^2n(n+1)(1-q)^{-n-2} \\
&= \frac{n(1-p)}{p^2}.
\end{aligned}$$

Whew! And that was not even close to some other calculations I have done over the years.

References

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[5] S. Ross, *A first course in probability*, Macmillan Co., New York, second ed., 1984.

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