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Lecture 8: Continuous Time Markov Chains(CTMC)

Standard texts on Continuous Time Markov Chains Dont know yet [].

1 Preliminaries

1.1 Definition

Unofficially, Continous Time Markov Chain is basically the same as Discrete Time Markov Chains.In Continuous Time Markov Chains we will also be considering the amount of time spent in each state. Mathematically speaking

$\{X_t\}_{(t \geq 0)}$ is a continous markov chain if the following happens

$$
\begin{aligned}
P[X_{(t+s)} = j | X_s = i, X_u = x(u), u < s] &= P[X_{(t+s)} = j | X_s = i] \\
&= P[X_t = j | X_0 = i] \\
p(t) &= (p_{ij}(t))_{(i,j) \in I}
\end{aligned}
$$

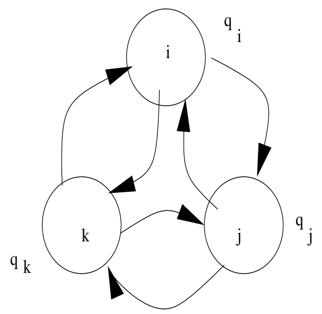

Figure 1: q_i, q_2, q_3 are amount of time spent in each state

In Continous Time Markov Chains we have infinite no. of transition matrices with repsect to time.

1.2 Example Processes of Continous Time Markov Chains

Some exapmle processes where Continous Time Markov Chains are applicable are given in this section. In all these processes when we say continous we preclude the fact that the two events can happen simultaneuosly ie. the time of arrival and leaning are not the same.

1.Birth and death process:
In a population of n people every individual dies with a certain probability, We can say that the rate of

arrival to be the birth rate $\{\lambda_n\}_{n\geq 0}$ and the rate of leaving to be the death rate $\{\mu_n\}_{n\geq 1}$(this is beacause there cannot be any deaths with no people,therefore $n \neq 0$).

2.M/M/1 Queueing System:
Consider an Internet server that recieves packets from a queue of packets.This process is a continous process as packets can come at any time and we cannot say that the no. of packets is $N(t)$, and t can take any value . The rate of arrival in the queue is said to be λ.The average amount of time spent by a packet in the server is $1/\mu$.

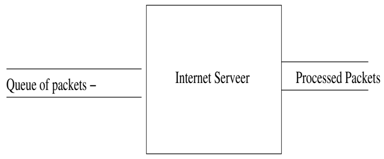

Figure 2: Single server

Let $q_0, q_1, \cdots$ be the amount of time spent in each state. The probability to move from state 0 to state1 is given as $p_{01} = 1$.This is beacause there is no one to die.

Figure 3: $q_0, q_1, q_2 \cdots$ are amount of time spent in each state

$$q_i = \frac{1}{E(\text{time before either a birth or a death occurs})}$$

These can be considered as 2 poisson processes based on different rates.One being the process of arrival(birth) and the other the process of leaving(death)

$$p_{i,i+1} = \frac{\lambda_i}{\mu_i + \lambda_i}$$
$$p_{i,i-1} = \frac{\mu_i}{\mu_i + \lambda_i}$$

Note:- Let $T_1, T_2, \cdots, T_n$ be exponential variables with rates $\lambda_1, \lambda_2 \cdots \lambda_n$

$$
\begin{aligned}
z &= min(T_1, T_2 \cdots T_n) \\
E[z] &= \frac{1}{\lambda_1 + \lambda_2 \cdots \lambda_n} \\
q_i &= \lambda_i + \mu_i
\end{aligned}
$$

A speacial case of the poisson process is the Birth and death process where

$$
\begin{aligned}
\lambda_n &= \lambda \\
\mu_n &= 0
\end{aligned}
$$

3.Yule Process (Pure Birth process) This a process where there is only arrival rate and no leaving rate.There is no death rate.

$$
\begin{aligned}
\lambda &= n\lambda \rightarrow \text{Every person gives birth independently at the rate } \lambda \\
\mu_n &= 0
\end{aligned}
$$

4.Linear growth with immigration.

$$
\begin{aligned}
\mu_n &= n\mu \rightarrow \text{(This implies every person dies independent of the rate } \mu) \\
\lambda_n &= n\lambda + \theta \text{(here } \theta \text{ is the immigration constant)}
\end{aligned}
$$

5.M/M/S Queueing system

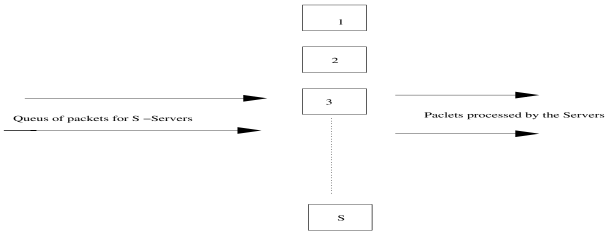

Figure 4: Multiple servers

There is a queue of packets that are given to S servers. The arrival rate is

$$
\begin{aligned}
\lambda_n &= \lambda \\
\mu_n &= n\mu \quad \text{if } n \leq S \\
&= S\mu \quad \text{otherwise(server's hands are full)}
\end{aligned}
$$

$P_{ij}(t)$ - This is the quantity of interest.That is at time t what is the probaability of having from $i \to j$ packets.Now we will calculate p_{ij} for some of the Continous Time Markov Chains examples given above

Ina pure birth process

$$\begin{aligned}
\lambda_n &\geq 0 \\
\mu_n &= 0 \\
p_{ij}(t) &= P[X_t = j|X_0 = i] \\
p_{ij} &\to 0 \quad \text{if } i > j
\end{aligned}$$

Hence,

$$\begin{aligned}
P[X_t < j|X_0 = i] &= P[T_i \cdots T_{j-1} > t] \\
P[X_t < j|X_0 = i] &= P[T_1 + T_2 \cdots T_{j-1} > t] \\
&= \sum_{k=0}^{j-1} e^{-\lambda k t} \Pi_{r=i,r\neq k}^{j-1} \frac{\lambda_r}{\lambda_r - \lambda_k}
\end{aligned}$$

We have the above from the following

$$\begin{aligned}
P[T_1 + T_2 > t] \to &= \int_0^t P[T_1 = s]P[T_2 = t - s]ds + \int_t^{+\infty} P[T_1 = s]ds \\
\int_0^t \lambda_1 e^{-\lambda_1 s} e^{-\lambda_2(t-s)} ds + e^{-\lambda_2 t} &= \lambda_1 e^{-\lambda_2 t} \int_0^t e^{-(\lambda_1 - \lambda_2)s} ds + e^{-\lambda_2 t} \\
\frac{\lambda_1 e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}[1 - e^{-(\lambda_1 - \lambda_2)t}] + e^{-\lambda_2 t} &= \frac{\lambda_1 . e^{-\lambda_2 t}}{\lambda_1 - \lambda_2} + \frac{\lambda_2 . e^{-\lambda_2 t}}{\lambda_1 - \lambda_2}
\end{aligned}$$

$$\begin{aligned}
P_{i,i+2}(t) &= P[T + T_{i+1} > t] \\
P[X_t = i + 2|X_0 = i] &= P[X_T < i + 3|X_0 =] - P[X_T < i + 2|X_0 = i] \\
p_{ij}(t) &= P_i(X_t < j + 1] - P_i[X_t < j]
\end{aligned}$$

Calculating $p_{ij}(t)$ Yules Process(Pure birth)

$$\begin{aligned}
\lambda_n = n\lambda \\
\mu_n = 0 \\
p_{ij}(t) = e^{\lambda t}(1 - e^{\lambda t})^{j-1}
\end{aligned}$$

$$
\begin{aligned}
P_{ij} = &\text{ Probability that the total no .of flips is equal to } j \text{ to get } i \text{ heads} \\
&= \sum_{n1+n2+\cdots} e^{-\lambda t_i}(1-e^{-\lambda t_i})^{j-1} \\
&= \begin{pmatrix} j-1 \\ i-1 \end{pmatrix} e^{-\lambda t_i}.(1-e^{-\lambda t_i})^{j-1}
\end{aligned}
$$

M/M/S with Balking:
Let A_n be the probability that a new customer enters the queue given that there are n people in the system

1.3 Chapman Kolmogorov equation for Continuous Time Markov Chain

It is given as

$$
p_{ij} = \sum_k p_{ik}^{(s)}.p_{kj}^{(t)}
$$

The rate of transition from $i \to j$ is defined by

$$
q_{ij} = \lim_{n \to 0} \frac{p_{ij}^h}{h} = p_{ij}^{'}(0)
$$

This is know as the “jumping rate”.

For a poisonn process

$$
q_{i,i+1} = \lambda
$$

For pure Pure birth we know

$$
\begin{aligned}
\lambda &= m\lambda, \mu = 0 \\
q_{n,n+1} &= \lambda_n
\end{aligned}
$$