Lecture 9: Continuous Time Markov Chains

1 Continuous Time Markov Chain

1.1 Last Class:

Rate of transition from i to j

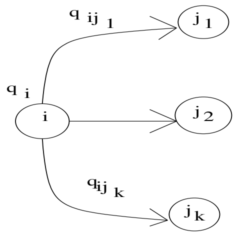


Figure 1: $q_{ij_1}...q_{ij_k}$ are amount of time spent in each state

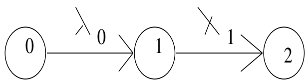


Figure 2: stays at 0 with rate λ_0 then goes to 1, stays at λ_1

$(X_t)_{t\geq 0} p(t) = p_{ij}(t)$

$q_{ij}(t) = \lim_{t\to 0} \frac{p_{ij}(t)}{t\to 0}$

Fraction of time for the transition is

$$q_i = \sum_{k\neq i} q_{ik}$$

which can also be defined as, rate of leaving i

1.2 Birth and Death Process:

If a population has n people then

λ_n : birth rate

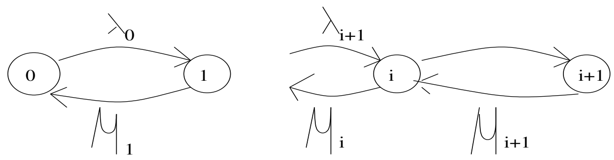


Figure 3: birth rate and death rate

μ_n : death rate (defined as time until next death exponentially)

T_i : length of time until process leaves i

T_i : min of {next death & next birth}

$$\begin{aligned} T_i &: exp(\lambda_i + \mu_i) \\ &= min\{exp(\lambda_i), exp(\mu_i)\} \end{aligned}$$

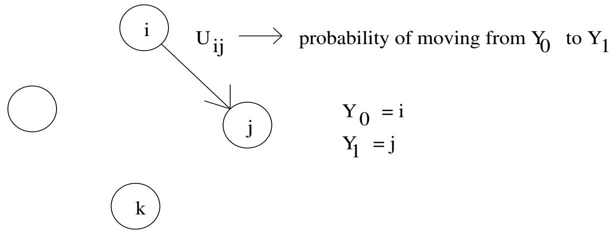
1.3 Example of Continuous Time Markov Chains:

$(Y_n)_{n\geq 0}$: Discrete Time Markov Chains with transition matrix

$U = (U_{ij})$

$N(t)$ is a Poisson process (λ) with distribution $(0 \leq \lambda \leq \infty)$ (this is also the number of arrivals)

To get Continuous Time Markov Chain from Discrete Time Markov Chain:



$X_t = Y_{N(t)}$

The significance is that, every time there is an arrival, X moves to another state.
To Calculate p_{ij}:

$$\begin{aligned} p_{ij} &= \sum_{n=0}^{\infty}(P[N(t)=n])U_{ij}^{(n)} \\ &= e^{-\lambda t}\frac{(\lambda t)^n}{n!}U_{ij}^{(n)} \end{aligned}$$

where $U_{ij}^{(n)}$ is the probability of taking n steps to go from i to j

1.4 Relation between p_{ij} and q:

Forward Kolmogorov Differential Equation:

$$\begin{aligned} p_{ij}(t+h) - p_{ij}(t) &= \sum_{k\in I} p_{ik}(t)p_{kj}(h) - p_{ij}(t) \\ &= \sum_{k\neq j} p_{ik}(t)p_{kj}(h) + p_{ij}(t)[p_{jj}(h)-1] \\ &= \lim_{h\to 0}\sum_{k\neq j}\frac{p_{ik}(t)}{h}p_{kj}(h) + \lim_{h\to 0}\frac{p_{ij}(t)}{h}[p_{jj}(h)-1] \end{aligned}$$

$$\begin{aligned} p_{ij}^{'}(t) &= \sum_{k\neq j} p_{ik}(t)\lim_{h\to 0}\frac{p_{kj}(h)}{h} - p_{ij}(t)\lim_{h\to 0}(\frac{1-p_{jj}(h)}{h}) \\ &= \sum_{k\neq j} p_{ik}(t)q_{kj} - p_{ij}(t)\sum_{k\neq j} q_{jk} \end{aligned}$$

Let $q_{jj} = -q_j$

Then

$p_{ij}{}'(t) = \sum_{k\in I} p_{ik}(t)q_{kj}$
$=> [P(t)]' = P(t).Q$ – Forward Kolmogorov Differential Equation

Similarly,

$=> [P(t)]' = Q.P(t)$ – Reverse Kolmogorov Differential Equation

Solving Matrix Equations:

If Q were a matrix
$P(t) = e^{Qt} \equiv \sum_{n=0}^{\infty} Q^n \frac{t^n}{n!}$

Inferences:

- Existence of a solution for $P(t)$

- Used for approximating $P(t)$
- If Q is diagonalizable then

$Q = M\Lambda M^{-1}$

$$\begin{aligned}
P(t) &= \sum M\Lambda^n M^{-1}\frac{t^n}{n!} \\
&= M[\sum \Lambda^n \frac{t^n}{n!}] \\
&= M| \begin{matrix} e^{\lambda_1 t}.. \\ .e^{\lambda_2 t}. \\ ... \\ ..e^{\lambda_n t} \end{matrix} |
\end{aligned}$$

1.5 More examples: Working behavior of Machines

Machines work: $\frac{1}{\lambda}$
Time it takes to fix: $\frac{1}{\mu}$

Question: What is the probability that the machine will work some years later?

$$\begin{aligned}
p_{11}^{'}(t) &= \lambda p_{21}(t) - \lambda p_{11}(t) && (1) \\
p_{21}^{'}(t) &= \mu p_{11}(t) - \mu p_{21}(t) && (2)
\end{aligned}$$

Solving Equations (1) & (2)

$$\begin{aligned}
\mu p_{11}^{'}(t) + \lambda p_{21}^{'}(t) &= 0 \\
\mu p_{11}(t) + \lambda p_{21}(t) &= C
\end{aligned}$$

where $C = \mu$ since $p_{11}(0) = 1 \& p_{21}(0) = 0$

$$\begin{aligned}
\mu p_{11}(t) + \lambda p_{21}(t) &= \mu \\
p_{11}^{'}(t) &= \mu - (\mu + \lambda)p_{11}(t)
\end{aligned}$$

Let $h(t)$ = RHS

$$\begin{aligned}
h^{'}(t) &= -(\mu + \lambda).LHS \\
h^{'}(t) &= -(\mu + \lambda).h(t) \\
h(t) &= ke^{-(\mu+\lambda)t} \\
&= \mu - (\mu + \lambda)p_{11}(t)
\end{aligned}$$

When $t = 0, k = -\lambda$
$=> p_{11}(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t}$

1.6 Convergence

Discrete Time Markov Chains converge when it is irreducible, aperiodic and have a stationary distribution.

Continuous Time Markov Chains converge when it is irreducible and have a stationary distribution.

Whats Irreducible?

X_t is irreducible iff $\forall i, j \in I, \exists i_1, i_2....i_m, s.t.q_{ii_1}, q_{i_1 i_2}.....q_{i_m j} > 0$ where q_{ij} are jumping rates

Theorem 1: If a Continuous Time Markov Chain is irreducible and has a stationary distribution π then $\lim t \to \infty p_{it}(t) = \pi_j$

Theorem 2: A distribution is invariant iff $\pi Q = 0$

$\forall i \exists I, \sum_{i \exists I} \pi_i q_{ik} = 0$

$\sum_{i \neq k} \pi_i q_{ik} = q_k \pi_k$ where $q_{kk} = -q_k$

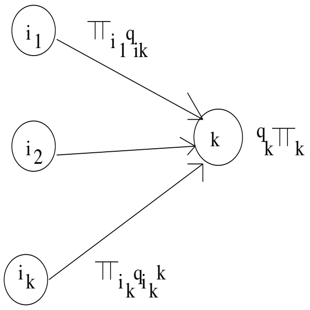


Figure 4: π_k is fraction of time spent in k

Rate of moving in = Rate of moving out.

Example: Birth and Death Process:

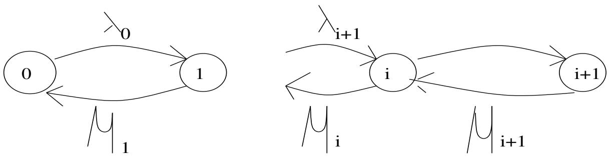


Figure 5: birth rate and death rate

State	Enter Rate	Exit Rate
0	$\mu_1\pi_1$	$\lambda_0\pi_0$
1	$\lambda_0\pi_0 + \mu_2\pi_2$	$(\lambda_1 + \mu_1)\pi_1$
.	.	.
.	.	.
i	$\lambda_{i-1}\pi_{i-1} + \mu_{i+1}\pi_{i+1}$	$\pi_i(\lambda_i + \mu_i)$

$$\begin{aligned}
\pi_1 &= \frac{\lambda_0}{\mu_1}\pi_0 \\
\pi_2 &= \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi_0 \\
\pi_n &= \frac{\lambda_0\lambda_1......\lambda_{n-1}}{\mu_1\mu_2.....\mu_n}\pi_0
\end{aligned}$$

If sum of all $\pi' s = 1$ then it is a distribution

$S = (1 + \frac{\lambda_0}{\mu_1} + ... + \frac{\lambda_0\lambda_1......\lambda_{n-1}}{\mu_1\mu_2.....\mu_n} + ..)$

If $S < \infty$ then $\pi_0 = \frac{1}{S}$

$\pi_n = \frac{\lambda_0.....\lambda_{n-1}}{\mu_1....\mu_n}\frac{1}{S}$

$\lim_{t\to\infty} p_{in}(t) = \pi_n$

If $S = \infty$ then, the process is transient.

1.7 Instance of Birth & Death: M $\setminus$ M $\setminus$ S

$\lambda_n = \lambda$
$\mu_n = min(n, s)\mu$

$$\begin{aligned}
S &= 1 + \sum_{n=1}^{\infty} \frac{\lambda_0.....\lambda_{n-1}}{\mu_1....\mu_n} \\
&= 1 + \sum_{n=1}^{s} \frac{\lambda^n}{n!\mu^n} + \sum_{n=1}^{\infty} (\frac{\lambda}{s\mu})^n (\frac{\lambda^s}{s!\mu^s})
\end{aligned}$$

The 1st term is a constant. The second term converges iff $\lambda < s\mu$