

CSE 694 Homework Assignment 1

Due in class on Tuesday, Feb 01

There are totally 8 problems, 10 points each. This assignment is meant to refresh your memory of basic probability theory. Feel free to consult textbooks and/or the web; however, if you do so, you have to tell me where you get the solution from. You are also encouraged to discuss problems with your classmates. Again, tell me who you talk to on which problem. The final writeup must still be your own, in your own words!

Please come talk to me if you have trouble doing this assignment. Understanding the basics is crucial before we go further in the course.

Below are a few theorems which could be of some use. These are the ones I did not mention specifically in the lecture notes.

Theorem 1 (Inclusion-Exclusion Principle). *Let A_1, \dots, A_n be n events. Then,*

$$\text{Prob} \left[\bigcup_{i=1}^n A_i \right] = \sum_{i=1}^n \text{Prob}[A_i] - \sum_{i < j} \text{Prob}[A_i \cap A_j] + \sum_{i < j < k} \text{Prob}[A_i \cap A_j \cap A_k] - \dots + (-1)^{n+1} \text{Prob}[A_1 \cap \dots \cap A_n]$$

Theorem 2 (Bayes' Formula). *Suppose B_1, \dots, B_n are mutually exclusive events of a probability space Ω (i.e. $B_i \cap B_j = \emptyset$ whenever $i \neq j$ and $\bigcup_i B_i = \Omega$.) (We also say the events B_1, \dots, B_n partition the sample space.) Then, the law of total probability states that*

$$\text{Prob}[A] = \sum_i \text{Prob}[A \cap B_i] = \sum_i \text{Prob}[A|B_i] \text{Prob}[B_i] \quad (1)$$

(In words, $\text{Prob}[A]$ is the weighted average of the $\text{Prob}[A|B_i]$.) The equation leads to what's called the Bayes formula:

$$\text{Prob}[B_j|A] = \frac{\text{Prob}[B_j \cap A]}{\text{Prob}[A]} = \frac{\text{Prob}[A|B_j] \text{Prob}[B_j]}{\sum_i \text{Prob}[A|B_i] \text{Prob}[B_i]} \quad (2)$$

Problem 1. In multiple choice tests, assume a student knows the answer to any question with probability p (i.e. she take a random guess with probability $1 - p$). Let m be the number of alternatives for each question. What is the probability that she knew the answer to a question given that she answered it correctly? What is the sample space in this example?

Problem 2. Alice and Bob go target shooting. Each of Alice's shots hits the target with probability $1/3$. Each of Bob's shots hits the target with probability $1/4$. They take turn shooting, with Alice shooting first. Determine

- The probability that the first shot on target from either of them is Alice's third shot
- The probability that Bob will hit the target before Alice does.

Problem 3. Suppose a fair coin is tossed until a head is obtained, and that this experiment is performed independently a second time. What is the probability that the second experiment requires more tosses than the first experiment?

Problem 4. Suppose a fair coin is tossed independently n times. Determine the probability of obtaining exactly $n - 1$ heads, given

- (a) that at least $n - 2$ heads are obtained,
- (b) heads are obtained on the first $n - 2$ tosses.

Problem 5. Suppose n guests check their hats in as they arrive at a restaurant, and that these hats are returned to them in a totally random order when they leave. Let $k \leq n$ be a given non-negative integer. Let X be the number of guests who get back their own hats. Determine

- (a) the probability $\text{Prob}[X = k]$
- (b) the expectation $E[X]$
- (c) the variance $\text{Var}[X]$

Problem 6. Suppose I roll 5 fair dice. What is the probability that their sum is divisible by 6? (Hint: apply the law of total probability, conditioned on the sum of the first 4 dice.)

Problem 7. A fair coin is tossed until there are k consecutive heads. What is the expected number of tosses?

Problem 8. Tom is torn between two lovers Alice and Barb. The three of them live on the same street, where Tom's house is in between Alice's and Barb's houses. Buses going the Alice to Barb direction and Barb to Alice direction at the exact same speed and regularity (say, an hour each).

Tom decides to randomly go to the bus station at his house, and take the first available bus. If the bus goes to Barb's direction, he'd to Barb's house, and vice versa.

After a long period of time, Tom realizes that he spends 3 times more at Barb's place than at Alice's place. How is that possible?