Overview and Plan

- Covering Chapter 2 of DHS (in two classes).
- Bayesian Decision Theory is a fundamental statistical approach to the problem of pattern classification.
- Quantifies the tradeoffs between various classifications using probability and the costs that accompany such classifications.
- Assumptions:
  - Decision problem is posed in probabilistic terms.
  - All relevant probability values are known.
Recall the Fish!

- Recall our example from the first lecture on classifying two fish as salmon or sea bass.
- And recall our agreement that any given fish is either a salmon or a sea bass; DHS call this the state of nature of the fish.
- Let’s define a (probabilistic) variable $\omega$ that describes the state of nature.

$$\omega = \omega_1 \quad \text{for sea bass} \quad (1)$$

$$\omega = \omega_2 \quad \text{for salmon} \quad (2)$$
Prior Probability

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- In the fish example, it is the probability that we will see either a salmon or a sea bass next on the conveyor belt.
- Note: The prior may vary depending on the situation.
  - If we get equal numbers of salmon and sea bass in a catch, then the priors are equal, or *uniform*.
  - Depending on the season, we may get more salmon than sea bass, for example.
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We write $P(\omega = \omega_1)$ or just $P(\omega_1)$ for the prior the next is a sea bass.

The priors must exclusivity and exhaustivity. For $c$ states of nature, or classes:

$$1 = \sum_{i=1}^{c} P(\omega_i)$$ (3)
Idea Check: What is a reasonable Decision Rule if
- The only available information is the prior.
- The cost of any incorrect classification is equal.
**Decision Rule From Only Priors**

- **Idea Check:** What is a reasonable Decision Rule if
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- Decide $\omega_1$ if $P(\omega_1) > P(\omega_2)$; otherwise decide $\omega_2$.
- What can we say about this decision rule?
**Decision Rule From Only Priors**

- **Idea Check**: What is a reasonable Decision Rule if
  - The only available information is the prior.
  - The cost of any incorrect classification is equal.

- Decide $\omega_1$ if $P(\omega_1) > P(\omega_2)$; otherwise decide $\omega_2$.

- What can we say about this decision rule?
  - Seems reasonable, but it will **always** choose the same fish.
  - If the priors are uniform, this rule will behave poorly.
  - Under the given assumptions, no other rule can do better! (We will see this later on.)
Features and Feature Spaces

- A **feature** is an observable variable.
- A **feature space** is a set from which we can sample or observe values.
- Features:
  - Length
  - Width
  - Lightness
  - Location of Dorsal Fin
- For simplicity, let’s assume that our features are all continuous values.
- Denote a scalar feature as $x$ and a vector feature as $\mathbf{x}$. For a $d$-dimensional feature space, $\mathbf{x} \in \mathbb{R}^d$. 

A note on the use of the term marginals as features (from first lecture): technically, a marginal is a distribution of one or more variables (e.g., $p(x)$). So, during modeling, when we say a “feature” is like a marginal, we are actually saying “the distribution of a type of feature” is like a marginal. This is only for conceptual reasoning.
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The class-conditional probability density function is the probability density function for \( x \), our feature, given that the state of nature is \( \omega \):

\[
p(x|\omega)
\]  

Here is the hypothetical class-conditional density \( p(x|\omega) \) for lightness values of sea bass and salmon.
Posterior Probability
Bayes Formula

- If we know the prior distribution and the class-conditional density, how does this affect our decision rule?
- **Posterior probability** is the probability of a certain state of nature given our observables: \( P(\omega|\mathbf{x}) \).
- Use Bayes Formula:

\[
P(\omega, \mathbf{x}) = P(\omega|\mathbf{x})p(\mathbf{x}) = p(\mathbf{x}|\omega)P(\omega) \tag{5}
\]

\[
P(\omega|\mathbf{x}) = \frac{p(\mathbf{x}|\omega)P(\omega)}{p(\mathbf{x})} \tag{6}
\]

\[
= \frac{p(\mathbf{x}|\omega)P(\omega)}{\sum_i p(\mathbf{x}|\omega_i)P(\omega_i)} \tag{7}
\]
Notice the likelihood and the prior govern the posterior. The $p(x)$ evidence term is a scale-factor to normalize the density.

For the case of $P(\omega_1) = 2/3$ and $P(\omega_2) = 1/3$ the posterior is
For a given observation $x$, we would be inclined to let the posterior govern our decision:

$$\omega^* = \arg \max_i P(\omega_i | x)$$  \hspace{1cm} (8)

What is our probability of error?
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\[
\omega^* = \arg \max_i P(\omega_i | x)
\]  

(8)

What is our probability of error?

For the two class situation, we have

\[
P(\text{error} | x) = \begin{cases} 
P(\omega_1 | x) & \text{if we decide } \omega_2 \\ 
P(\omega_2 | x) & \text{if we decide } \omega_1 \end{cases}
\]

(9)
We can minimize the probability of error by following the posterior:

\[
\text{Decide } \omega_1 \text{ if } P(\omega_1|x) > P(\omega_2|x) \quad (10)
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Decide $\omega_1$ if $P(\omega_1|x) > P(\omega_2|x)$  \hspace{1cm} (10)

And, this minimizes the average probability of error too:

$$P(\text{error}) = \int_{-\infty}^{\infty} P(\text{error}|x)p(x)dx$$  \hspace{1cm} (11)

(Because the integral will be minimized when we can ensure each $P(\text{error}|x)$ is as small as possible.)
Bayes Decision Rule (with Equal Costs)

- Decide $\omega_1$ if $P(\omega_1|\mathbf{x}) > P(\omega_2|\mathbf{x})$; otherwise decide $\omega_2$
- Probability of error becomes

$$P(\text{error}|\mathbf{x}) = \min[P(\omega_1|\mathbf{x}), P(\omega_2|\mathbf{x})] \quad (12)$$
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P(\text{error}|x) = \min [P(\omega_1|x), P(\omega_2|x)]
\] (12)

- Equivalently, Decide \( \omega_1 \) if \( p(x|\omega_1)P(\omega_1) > p(x|\omega_2)P(\omega_2) \); otherwise decide \( \omega_2 \)
- I.e., the evidence term is not used in decision making.
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- I.e., the evidence term is not used in decision making.
- If we have $p(x|\omega_1) = p(x|\omega_2)$, then the decision will rely exclusively on the priors.
- Conversely, if we have uniform priors, then the decision will rely exclusively on the likelihoods.
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- Conversely, if we have uniform priors, then the decision will rely exclusively on the likelihoods.
- Take Home Message: Decision making relies on both the priors and the likelihoods and Bayes Decision Rule combines them to achieve the minimum probability of error.
A **loss function** states exactly how costly each action is.

As earlier, we have \( c \) classes \( \{\omega_1, \ldots, \omega_c\} \).

We also have \( a \) possible actions \( \{\alpha_1, \ldots, \alpha_a\} \).

The loss function \( \lambda(\alpha_i|\omega_j) \) is the loss incurred for taking action \( \alpha_i \) when the class is \( \omega_j \).
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The loss function \( \lambda(\alpha_i | \omega_j) \) is the loss incurred for taking action \( \alpha_i \) when the class is \( \omega_j \).

The Zero-One Loss Function is a particularly common one:

\[
\lambda(\alpha_i | \omega_j) = \begin{cases} 
0 & i = j \\
1 & i \neq j 
\end{cases} \quad i, j = 1, 2, \ldots, c
\]  

(13)

It assigns no loss to a correct decision and uniform unit loss to an incorrect decision. (Similar to Dirac delta function...)
Expected Loss  
aka. Conditional Risk

- We can consider the loss that would be incurred from taking each possible action in our set.
- The **expected loss** is by definition

\[ R(\alpha_i|x) = \sum_{j=1}^{c} \lambda(\alpha_i|\omega_j)P(\omega_j|x) \]  \hspace{1cm} (14)

- The **zero-one conditional risk** is

\[ R(\alpha_i|x) = \sum_{j \neq i} P(\omega_j|x) \]  \hspace{1cm} (15)

\[ = 1 - P(\omega_i|x) \]  \hspace{1cm} (16)

- Hence, for an observation \( x \), we can minimize the expected loss by selecting the action that minimizes the conditional risk.
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Hence, for an observation $\mathbf{x}$, we can minimize the expected loss by selecting the action that minimizes the conditional risk.

(Teaser) You guessed it: this is what Bayes Decision Rule does!
Let $\alpha(x)$ denote a decision rule, a mapping from the input feature space to an action, $\mathbb{R}^d \mapsto \{\alpha_1, \ldots, \alpha_a\}$.

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- This is what we want to learn.

The **overall risk** is the expected loss associated with a given decision rule.

$$R = \int R(\alpha(x)|x) p(x) \, dx$$ (17)

Clearly, we want the rule $\alpha(\cdot)$ that minimizes $R(\alpha(x)|x)$ for all $x$. 
Bayes Decision Rule gives us a method for minimizing the overall risk.

Select the action that minimizes the conditional risk:

$$\alpha^* = \arg \min_{\alpha_i} R(\alpha_i | x)$$  \hspace{1cm} (18)

$$= \arg \min_{\alpha_i} \sum_{j=1}^{c} \lambda(\alpha_i | \omega_j) P(\omega_j | x)$$  \hspace{1cm} (19)

The Bayes Risk is the best we can do.
Two-Category Classification Examples

- Consider two classes and two actions, $\alpha_1$ when the true class is $\omega_1$ and $\alpha_2$ for $\omega_2$.
- Writing out the conditional risks gives:
  \[
  R(\alpha_1|x) = \lambda_{11}P(\omega_1|x) + \lambda_{12}P(\omega_2|x) \quad (20)
  \]
  \[
  R(\alpha_2|x) = \lambda_{21}P(\omega_1|x) + \lambda_{22}P(\omega_2|x) \quad (21)
  \]
- Fundamental rule is decide $\omega_1$ if
  \[
  R(\alpha_1|x) < R(\alpha_2|x) \quad (22)
  \]
- In terms of posteriors, decide $\omega_1$ if
  \[
  (\lambda_{21} - \lambda_{11})P(\omega_1|x) > (\lambda_{12} - \lambda_{22})P(\omega_2|x) \quad (23)
  \]
  The more likely state of nature is scaled by the differences in loss (which are generally positive).
Two-Category Classification Examples

- Or, expanding via Bayes Rule, decide $\omega_1$ if
  \[
  (\lambda_{21} - \lambda_{11})p(x|\omega_1)P(\omega_1) > (\lambda_{12} - \lambda_{22})p(x|\omega_2)P(\omega_2)
  \]  
  (24)

- Or, assuming $\lambda_{21} > \lambda_{11}$, decide $\omega_1$ if
  \[
  \frac{p(x|\omega_1)}{p(x|\omega_2)} = \frac{\lambda_{12} - \lambda_{22}}{\lambda_{21} - \lambda_{11}} \frac{P(\omega_2)}{P(\omega_1)}
  \]  
  (25)

- LHS is called the **likelihood ratio**.

- Thus, we can say the Bayes Decision Rule says to decide $\omega_1$ if the likelihood ratio exceeds a threshold that is independent of the observation $x$. 
Discriminant Functions are a useful way of representing pattern classifiers.

Let's say $g_i(x)$ is a discriminant function for the $i$th class.

This classifier will assign a class $\omega_i$ to the feature vector $x$ if

$$g_i(x) > g_j(x) \quad \forall j \neq i ,$$

or, equivalently

$$i^* = \arg \max_i g_i(x) , \quad \text{decide} \quad \omega_{i^*} .$$
We can view the discriminant classifier as a network (for \( c \) classes and a \( d \)-dimensional input vector).
Bayes Discriminants
Minimum Conditional Risk Discriminant

- General case with risks

\[ g_i(x) = -R(\alpha_i|x) \]  \hspace{1cm} (27)

\[ = - \sum_{j=1}^{c} \lambda(\alpha_i|\omega_j)P(\omega_j|x) \]  \hspace{1cm} (28)

- Can we prove that this is correct?
General case with risks

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Can we prove that this is correct?

Yes! The minimum conditional risk corresponds to the maximum discriminant.
In the case of zero-one loss function, the Bayes Discriminant can be further simplified:

\[ g_i(x) = P(\omega_i|x) \]  \hspace{1cm} (29)
Uniqueness Of Discriminants

- Is the choice of discriminant functions unique?
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- **No!**
- Multiply by some positive constant.
- Shift them by some additive constant.
Uniqueness Of Discriminants

- Is the choice of discriminant functions unique?
  - **No!**
  - Multiply by some positive constant.
  - Shift them by some additive constant.
  - For monotonically increasing function $f(\cdot)$, we can replace each $g_i(x)$ by $f(g_i(x))$ without affecting our classification accuracy.
    - These can help for ease of understanding or computability.
    - The following all yield the same exact classification results for minimum-error-rate classification.

\[
g_i(x) = P(\omega_i|x) = \frac{p(x|\omega_i)P(\omega_i)}{\sum_j p(x|\omega_j)P(\omega_j)} \quad (30)
\]

\[
g_i(x) = p(x|\omega_i)P(\omega_i) \quad (31)
\]

\[
g_i(x) = \ln p(x|\omega_i) + \ln P(\omega_i) \quad (32)
\]
The effect of any decision rule is to divide the feature space into decision regions.

Denote a decision region $R_i$ for $\omega_i$.

One not necessarily connected region is created for each category and assignments is according to:

$$\text{If } g_i(x) > g_j(x) \quad \forall j \neq i, \text{ then } x \text{ is in } R_i.$$  

Decision boundaries separate the regions; they are ties among the discriminant functions.
In this two-dimensional two-category classifier, the probability densities are Gaussian, the decision boundary consists of two hyperbolas, and thus the decision region $R_2$ is not simply connected. The ellipses mark where the density is $1/e$ times that at the peak of the distribution. From: Richard O. Duda, Peter E. Hart, and David G. Stork, Pattern Classification. Copyright © 2001 by John Wiley & Sons, Inc.
In the two-category case, one considers single discriminant

\[ g(x) = g_1(x) - g_2(x) . \]  \hspace{1cm} (34)

What is a suitable decision rule?
In the two-category case, one considers single discriminant

$$g(x) = g_1(x) - g_2(x).$$

(34)

The following simple rule is then used:

Decide $\omega_1$ if $g(x) > 0$; otherwise decide $\omega_2$.

(35)
Two-Category Discriminants

Dichotomizers

- In the two-category case, one considers single discriminant

\[ g(x) = g_1(x) - g_2(x) \]  \hspace{1cm} (34)

- The following simple rule is then used:

Decide \( \omega_1 \) if \( g(x) > 0 \); otherwise decide \( \omega_2 \). \hspace{1cm} (35)

- Various manipulations of the discriminant:

\[ g(x) = P(\omega_1|x) - P(\omega_2|x) \]  \hspace{1cm} (36)

\[ g(x) = \ln \frac{p(x|\omega_1)}{p(x|\omega_2)} + \ln \frac{P(\omega_1)}{P(\omega_2)} \]  \hspace{1cm} (37)
This next section is a slight digression to introduce the Normal Density (most of you will have had this already).

The Normal density is very well studied.

It is easy to work with analytically.

In many pattern recognition scenarios, an appropriate model seems to be where your data is assumed to be continuous-valued, randomly corrupted versions of a single typical value.

Central Limit Theorem (Second Fundamental Theorem of Probability).

- The distribution of the sum of $n$ random variables approaches the normal distribution when $n$ is large.
- E.g., http://www.statisticshowto.com/berrie/dsl/Galton.html
Recall the definition of expected value of any scalar function $f(x)$ in the continuous $p(x)$ and discrete $P(x)$ cases

$$E[f(x)] = \int_{-\infty}^{\infty} f(x)p(x)dx$$  \hspace{1cm} (38)

$$E[f(x)] = \sum_{x} f(x)P(x)$$  \hspace{1cm} (39)

where we have a set $D$ over which the discrete expectation is computed.
Continuous univariate normal, or **Gaussian**, density:

\[ p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right]. \tag{40} \]

**The mean** is the expected value of \( x \) is

\[ \mu \equiv \mathcal{E}[x] = \int_{-\infty}^{\infty} xp(x)dx. \tag{41} \]

**The variance** is the expected squared deviation

\[ \sigma^2 \equiv \mathcal{E}[(x - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 p(x)dx. \tag{42} \]
Samples from the normal density tend to cluster around the mean and be spread-out based on the variance.
The normal density is completely specified by the mean and the variance. These two are its **sufficient statistics**.

We thus abbreviate the equation for the normal density as

\[ p(x) \sim N(\mu, \sigma^2) \]  

\[ (43) \]
**Entropy**

- Entropy is the uncertainty in the random samples from a distribution.

\[ H(p(x)) = -\int p(x) \ln p(x) dx \]  \hspace{1cm} (44)

- The normal density has the maximum entropy for all distributions have a given mean and variance.

- What is the entropy of the uniform distribution?
Entropy

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- What is the entropy of the uniform distribution?

- The uniform distribution has maximum entropy (on a given interval).
The Normal Density
Multivariate Normal Density
And a test to see if your Linear Algebra is up to snuff.

The multivariate Gaussian in \( d \) dimensions is written as

\[
p(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det{\Sigma}}} \exp\left[-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right]. \tag{45}\]

Again, we abbreviate this as \( p(x) \sim N(\mu, \Sigma) \).

The sufficient statistics in \( d \)-dimensions:

\[
\mu \equiv \mathbb{E}[x] = \int x p(x) \, dx \tag{46}
\]

\[
\Sigma \equiv \mathbb{E}[(x - \mu)(x - \mu)^T] = \int (x - \mu)(x - \mu)^T p(x) \, dx \tag{47}
\]
The Covariance Matrix

\[ \Sigma \equiv \mathcal{E}[(x - \mu)(x - \mu)^T] = \int (x - \mu)(x - \mu)^T p(x) dx \]

- Symmetric.
- Positive semi-definite (but DHS only considers positive definite so that the determinant is strictly positive).
- The diagonal elements \( \sigma_{ii} \) are the variances of the respective coordinate \( x_i \).
- The off-diagonal elements \( \sigma_{ij} \) are the covariances of \( x_i \) and \( x_j \).
- What does a \( \sigma_{ij} = 0 \) imply?
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The Normal Density

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- What does \( \Sigma \) reduce to if all off-diagonals are 0?
The Covariance Matrix

\[ \Sigma \equiv \mathcal{E}[(\mathbf{x} - \mu)(\mathbf{x} - \mu)^T] = \int (\mathbf{x} - \mu)(\mathbf{x} - \mu)^T p(\mathbf{x}) d\mathbf{x} \]

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- What does a \( \sigma_{ij} = 0 \) imply?
- That coordinates \( x_i \) and \( x_j \) are statistically independent.
- What does \( \Sigma \) reduce to if all off-diagonals are 0?
- The product of the \( d \) univariate densities.
Linear Combinations of Normals

- Linear combinations of jointly normally distributed random variables, independent or not, are normally distributed.
- For \( p(x) \sim N((\mu), \Sigma) \) and \( A \), a \( d \)-by-\( k \) matrix, define \( y = A^T x \). Then:
  \[
  p(y) \sim N(A^T \mu, A^T \Sigma A)
  \] (48)
- With the covariance matrix, we can calculate the dispersion of the data in any direction or in any subspace.
The shape of the density is determined by the covariance $\Sigma$.

Specifically, the eigenvectors of $\Sigma$ give the principal axes of the hyperellipsoids and the eigenvalues determine the lengths of these axes.

The loci of points of constant density are hyperellipsoids with constant Mahalonobis distance:

$$\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu)$$  \hspace{1cm} (49)
Recall the minimum error rate discriminant,
\[ g_i(x) = \ln p(x|\omega_i) + \ln P(\omega_i). \]

If we assume normal densities, i.e., if \( p(x|\omega_i) \sim N(\mu_i, \Sigma_i) \), then the general discriminant is of the form
\[
g_i(x) = -\frac{1}{2}(x - \mu_i)^T \Sigma_i^{-1} (x - \mu_i) - \frac{d}{2} \ln 2\pi - \frac{1}{2} \ln |\Sigma_i| + \ln P(\omega_i)
\]
(50)
What do the decision boundaries look like if we assume $\Sigma_i = \sigma^2 I$?
Simple Case: Statistically Independent Features with Same Variance

- What do the decision boundaries look like if we assume $\Sigma_i = \sigma^2 I$?
- They are hyperplanes.

Let’s see why...
The Normal Density

**Simple Case:** $\Sigma_i = \sigma^2 I$

- The discriminant functions take on a simple form:

$$g_i(x) = -\frac{\|x - \mu_i\|^2}{2\sigma^2} + \ln P(\omega_i)$$  \hspace{1cm} (51)

- Think of this discriminant as a combination of two things
  1. The distance of the sample to the mean vector (for each $i$).
  2. A normalization by the variance and offset by the prior.
Simple Case: $\Sigma_i = \sigma^2 I$

- But, we don’t need to actually compute the distances.
- Expanding the quadratic form $(x - \mu)^T(x - \mu)$ yields

$$g_i(x) = -\frac{1}{2\sigma^2} \left[ x^T x - 2\mu_i^T x + \mu_i^T \mu_i \right] + \ln P(\omega_i) \quad (52)$$

- The quadratic term $x^T x$ is the same for all $i$ and can thus be ignored.
- This yields the equivalent **linear discriminant functions**

$$g_i(x) = w_i^T x + w_{i0} \quad (53)$$

$$w_i = \frac{1}{\sigma^2} \mu_i \quad (54)$$

$$w_{i0} = -\frac{1}{2\sigma^2} \mu_i^T \mu_i + \ln P(\omega_i) \quad (55)$$

- $w_{i0}$ is called the **bias**.
The Normal Density

Simple Case: $\Sigma_i = \sigma^2 I$

Decision Boundary Equation

- The decision surfaces for a linear discriminant classifiers are hyperplanes defined by the linear equations $g_i(x) = g_j(x)$.

- The equation can be written as

  $$w^T(x - x_0) = 0$$  \hspace{1cm} (56)
  $$w = \mu_i - \mu_j$$  \hspace{1cm} (57)
  $$x_0 = \frac{1}{2}(\mu_i + \mu_j) - \frac{\sigma^2}{\|\mu_i - \mu_j\|^2} \ln \frac{P(\omega_i)}{P(\omega_j)}(\mu_i - \mu_j)$$  \hspace{1cm} (58)

- These equations define a hyperplane through point $x_0$ with a normal vector $w$. 
Simple Case: $\Sigma_i = \sigma^2 I$

Decision Boundary Equation

- The decision boundary changes with the prior.
The discriminant functions are quadratic (the only term we can drop is the \(\ln 2\pi\) term):

\[
g_i(x) = x^T W_i x + w_i^T x + w_{i0}
\]  
(59)

\[
W_i = -\frac{1}{2} \Sigma_i^{-1}
\]  
(60)

\[
w_i = \Sigma_i^{-1} \mu_i
\]  
(61)

\[
w_{i0} = -\frac{1}{2} \mu_i^T \Sigma_i^{-1} \mu_i - \frac{1}{2} \ln|\Sigma_i| + \ln P(\omega_i)
\]  
(62)

- The decision surface between two categories are hyperquadratics.
General Case: Arbitrary $\Sigma_i$

FIGURE 2.14. Arbitrary Gaussian distributions lead to Bayes decision boundaries that are general hyperquadrics. Conversely, given any hyperquadric, one can find two Gaussian distributions whose Bayes decision boundary is that hyperquadric. These variances are indicated by the contours of constant probability density. From: Richard O. Duda, Peter E. Hart, and David G. Stork, Pattern Classification. Copyright © 2001 by John Wiley & Sons, Inc.
General Case: Arbitrary $\Sigma_i$
General Case for Multiple Categories

Quite A Complicated Decision Surface!
Signal Detection Theory

- A fundamental way of analyzing a classifier.
- Consider the following experimental setup:

Suppose we are interested in detecting a single pulse.
- We can read an internal signal $x$.
- The signal is distributed about mean $\mu_2$ when an external signal is present and around mean $\mu_1$ when no external signal is present.
- Assume the distributions have the same variances, $p(x|\omega_i) \sim N(\mu_i, \sigma^2)$. 

$$p(x|\omega_i)$$

$\omega_1$

$\sigma$

$\mu_1$

$x^*$

$\omega_2$

$\sigma$

$\mu_2$

$x$
The detector uses $x^*$ to decide if the external signal is present.

**Discriminability** characterizes how difficult it will be to decide if the external signal is present without knowing $x^*$.

\[ d' = \frac{|\mu_2 - \mu_1|}{\sigma} \]  

Even if we do not know $\mu_1$, $\mu_2$, $\sigma$, or $x^*$, we can find $d'$ by using a receiver operating characteristic or ROC curve.
A **Hit** is the probability that the internal signal is above $x^*$ given that the external signal is present

$$P(x > x^* | x \in \omega_2)$$  \hspace{1cm} (64)

- **Correct Rejection** is the probability that the internal signal is below $x^*$ given that the external signal is not present.

$$P(x < x^* | x \in \omega_1)$$  \hspace{1cm} (65)

- **False Alarm** is the probability that the internal signal is above $x^*$ despite there being no external signal present.

$$P(x > x^* | x \in \omega_1)$$  \hspace{1cm} (66)

- **Miss** is the probability that the internal signal is below $x^*$ given that the external signal is present.

$$P(x < x^* | x \in \omega_2)$$  \hspace{1cm} (67)
Receiver Operating Characteristics

Definitions

- A **Hit** is the probability that the internal signal is above $x^*$ given that the external signal is present
  \[ P(x > x^* | x \in \omega_2) \] (64)

- A **Correct Rejection** is the probability that the internal signal is below $x^*$ given that the external signal is not present.
  \[ P(x < x^* | x \in \omega_1) \] (65)
Receiver Operating Characteristics
Definitions

- A **Hit** is the probability that the internal signal is above $x^*$ given that the external signal is present.
  \[ P(x > x^* \mid x \in \omega_2) \]  

- A **Correct Rejection** is the probability that the internal signal is below $x^*$ given that the external signal is not present.
  \[ P(x < x^* \mid x \in \omega_1) \]  

- A **False Alarm** is the probability that the internal signal is above $x^*$ despite there being no external signal present.
  \[ P(x > x^* \mid x \in \omega_1) \]
Receiver Operating Characteristics

Definitions

- A **Hit** is the probability that the internal signal is above $x^*$ given that the external signal is present.
  \[ P(x > x^* | x \in \omega_2) \] (64)

- A **Correct Rejection** is the probability that the internal signal is below $x^*$ given that the external signal is not present.
  \[ P(x < x^* | x \in \omega_1) \] (65)

- A **False Alarm** is the probability that the internal signal is above $x^*$ despite there being no external signal present.
  \[ P(x > x^* | x \in \omega_1) \] (66)

- A **Miss** is the probability that the internal signal is below $x^*$ given that the external signal is present.
  \[ P(x < x^* | x \in \omega_2) \] (67)
We can experimentally determine the rates, in particular the Hit-Rate and the False-Alarm-Rate.

Basic idea is to assume our densities are fixed (reasonable) but vary our threshold $x^*$, which will thus change the rates.

The receiver operating characteristic plots the hit rate against the false alarm rate.

What shape curve do we want?
Missing Features

- Suppose we have built a classifier on multiple features, for example the lightness and width.
- What do we do if one of the features is not measurable for a particular case? For example the lightness can be measured but the width cannot because of occlusion.

Marginalize!

Let $x$ be our full feature set and $x_g$ be the subset that are measurable (or good) and let $x_b$ be the subset that are missing (or bad/noisy).

We seek an estimate of the posterior given just the good features $x_g$. 
Suppose we have built a classifier on multiple features, for example the lightness and width.

What do we do if one of the features is not measurable for a particular case? For example the lightness can be measured but the width cannot because of occlusion.

**Marginalize!**

Let \( x \) be our full feature feature and \( x_g \) be the subset that are measurable (or good) and let \( x_b \) be the subset that are missing (or bad/noisy).

We seek an estimate of the posterior given **just the good features** \( x_g \).
We will cover the Expectation-Maximization algorithm later.

This is normally quite expensive to evaluate unless the densities are special (like Gaussians).
Two variables $x_i$ and $x_j$ are independent if

$$p(x_i, x_j) = p(x_i) p(x_j)$$

(72)

**FIGURE 2.23.** A three-dimensional distribution which obeys $p(x_1, x_3) = p(x_1)p(x_3)$; thus here $x_1$ and $x_3$ are statistically independent but the other feature pairs are not. From: Richard O. Duda, Peter E. Hart, and David G. Stork, *Pattern Classification*. Copyright © 2001 by John Wiley & Sons, Inc.
An Early Graphical Model

- We represent these statistical dependencies graphically.
- Bayesian Belief Networks, or Bayes Nets, are directed acyclic graphs.
- Each link is directional.
- No loops.
- The Bayes Net factorizes the distribution into independent parts (making for more easily learned and computed terms).
Consider a simple example consisting of four variables: the weather, the presence of a cavity, the presence of a toothache, and the presence of other mouth-related variables such as dry mouth.

- The weather is clearly independent of the other three variables.
- And the toothache and catch are conditionally independent given the cavity (one as no effect on the other given the information about the cavity).

![Bayesian Belief Network Diagram]

Weather → Cavity → Toothache, Catch
Bayes Nets Components

- Each **node** represents one variable (assume discrete for simplicity).
- A **link** joining two nodes is directional and it represents **conditional probabilities**.
- The intuitive meaning of a link is that the source has a direct influence on the sink.
- Since we typically work with discrete distributions, we evaluate the conditional probability at each node given its parents and store it in a lookup table called a **conditional probability table**.

![Bayesian Belief Network Diagram](image-url)
Key: given knowledge of the values of some nodes in the network, we can apply Bayesian inference to determine the maximum posterior values of the unknown variables!
Consider a Bayes network with $n$ variables $x_1, \ldots, x_n$.

Denote the parents of a node $x_i$ as $\mathcal{P}(x_i)$.

Then, we can decompose the joint distribution into the product of conditionals

$$P(x_1, \ldots, x_n) = \prod_{i=1}^{n} P(x_i|\mathcal{P}(x_i))$$  \hspace{1cm} (73)
Belief at a Single Node

- What is the distribution at a single node, given the rest of the network and the evidence \( e \)?
- **Parents** of \( X \), the set \( \mathcal{P} \) are the nodes on which \( X \) is conditioned.
- **Children** of \( X \), the set \( \mathcal{C} \) are the nodes conditioned on \( X \).
- Use the Bayes Rule, for the case on the right:

\[
P(a, b, x, c, d) = P(a, b, x | c, d)P(c, d) \tag{74}
\]

\[
= P(a, b | x)P(x | c, d)P(c, d) \tag{75}
\]

or more generally,

\[
P(C(x), x, \mathcal{P}(x) | e) = P(C(x) | x, e)P(x | \mathcal{P}(x), e)P(\mathcal{P}(x) | e) \tag{76}
\]