

# Combinatorial Topology for Meshing

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There is an isomorphism  
between meshes and DAGs.

# Outline

## Meshes

Mesh Transformations

Future Work

## Hasse Diagram

Mesh Representation (Knepley and Karpeev 2009; Lange et al. 2016)  
for a CW-complex (Whitehead 1949)

DAG vertices  $\longleftrightarrow$   $k$ -cells

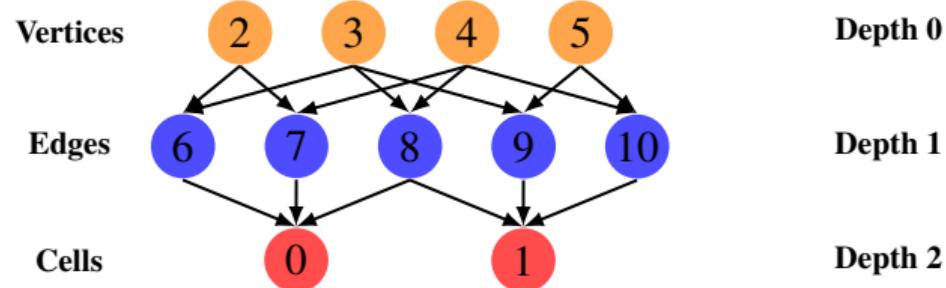
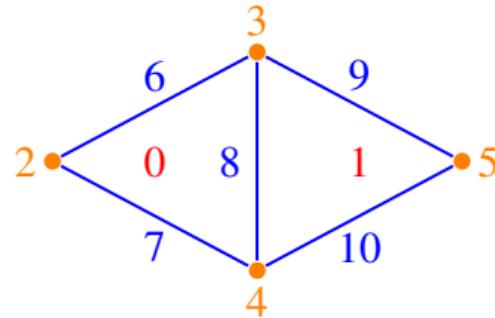
DAG edges  $\longleftrightarrow$  boundary adjacency

Edge labels  $\longleftrightarrow$  dihedral group representers

We call each DAG vertex a *mesh point*

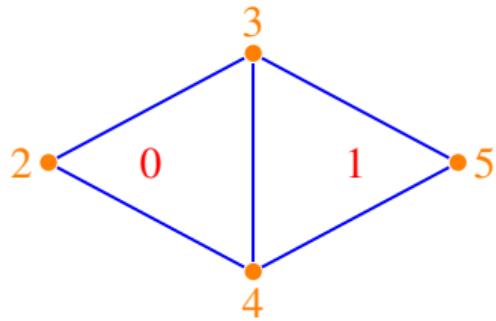
# Sample Meshes

Interpolated triangular mesh



# Sample Meshes

Optimized triangular mesh

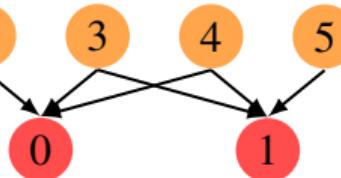


Vertices



Depth 0

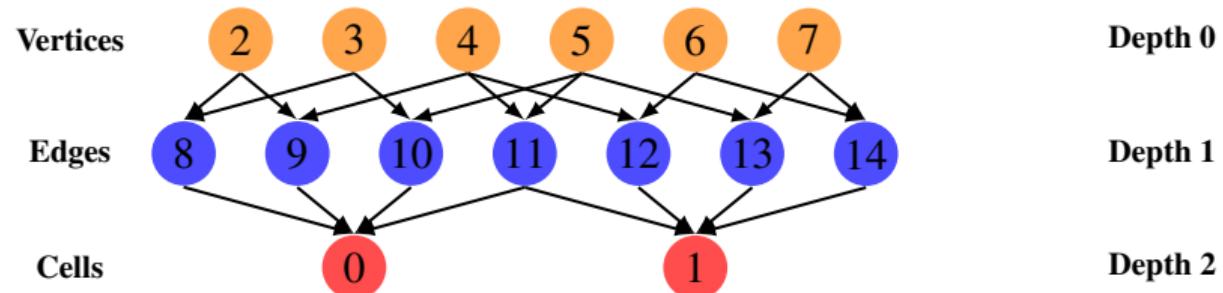
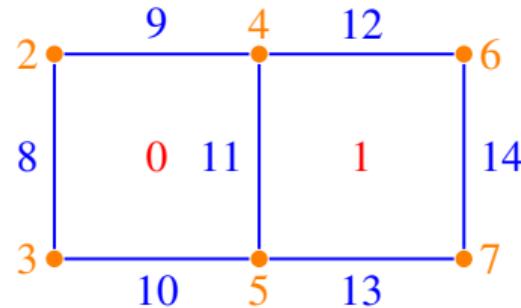
Cells



Depth 1

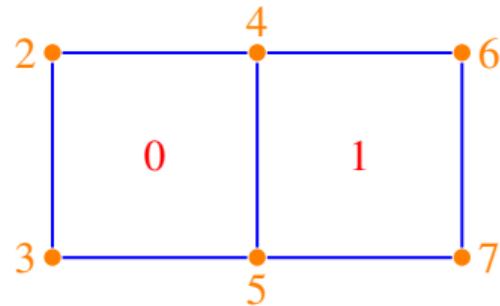
# Sample Meshes

Interpolated quadrilateral mesh

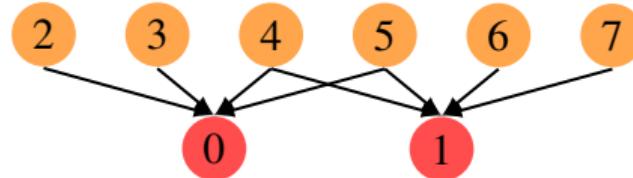


# Sample Meshes

Optimized quadrilateral mesh



Vertices



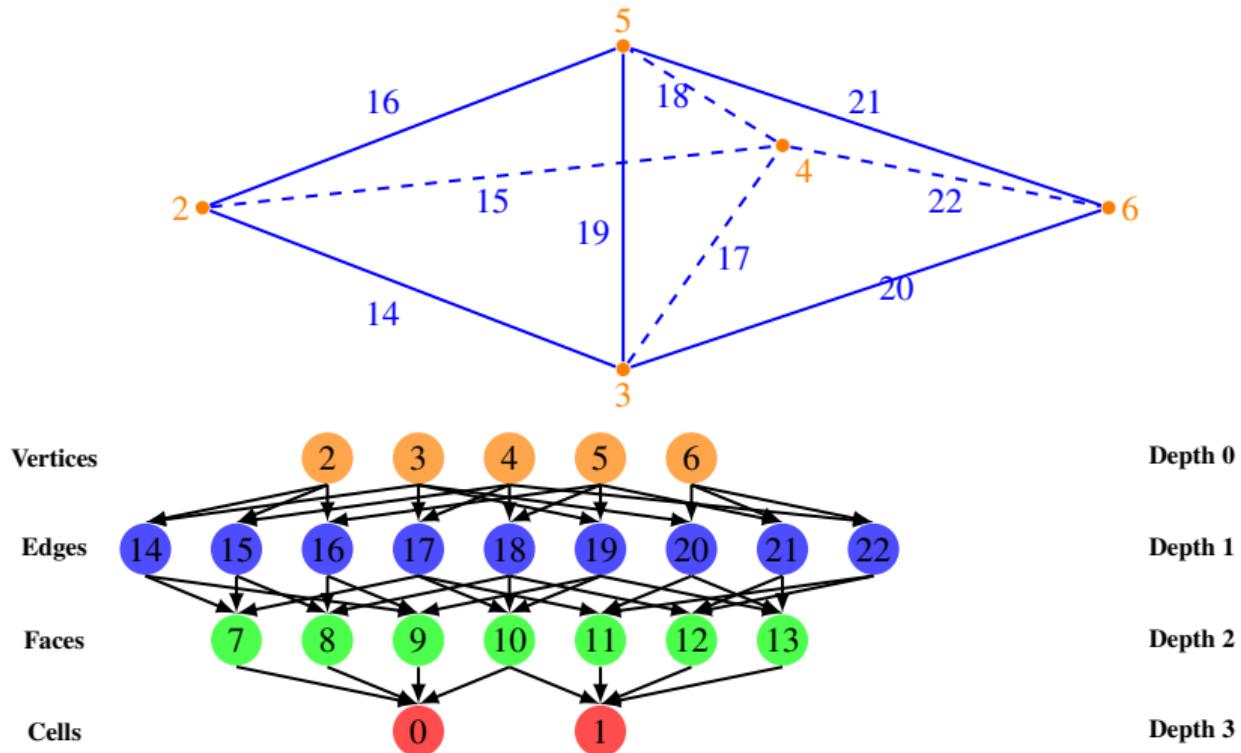
Depth 0

Cells

Depth 1

# Sample Meshes

Interpolated tetrahedral mesh

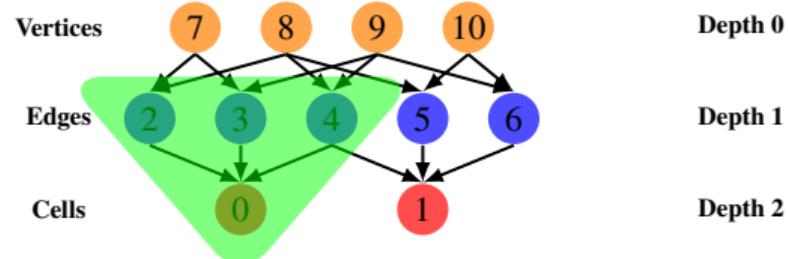
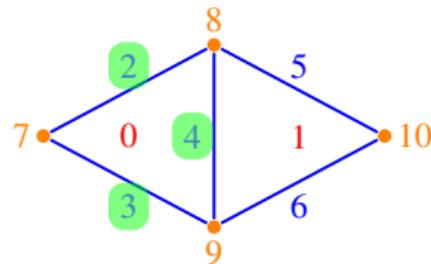


# Basic Operations

## Cone

We begin with the basic covering relation,

$$\text{cone}(0) = \{2, 3, 4\}$$

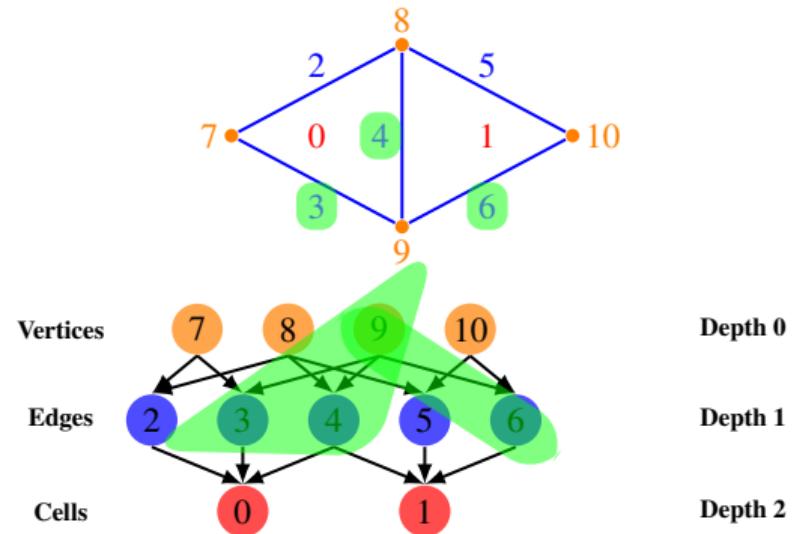


# Basic Operations

## Support

reverse arrows to get the dual operation,

$$\text{support}(9) = \{3, 4, 6\}$$

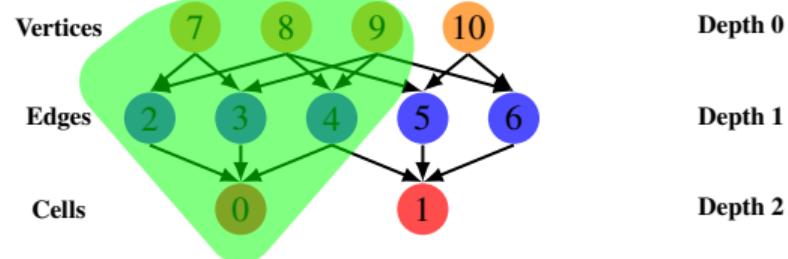
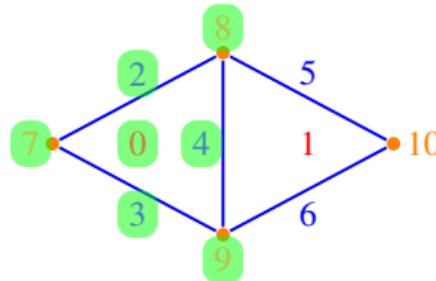


# Basic Operations

## Closure

add the transitive closure of  
the relation,

$$\text{closure}(0) = \{0, 2, 3, 4, 7, 8, 9\}$$

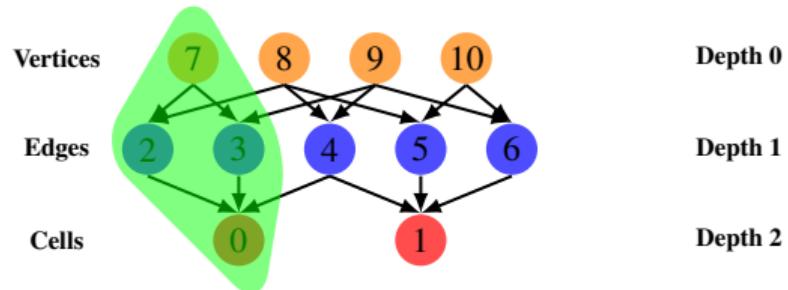
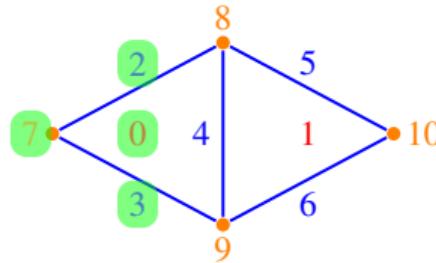


# Basic Operations

## Star

and the transitive closure of  
the dual,

$$\text{star}(7) = \{7, 2, 3, 0\}$$

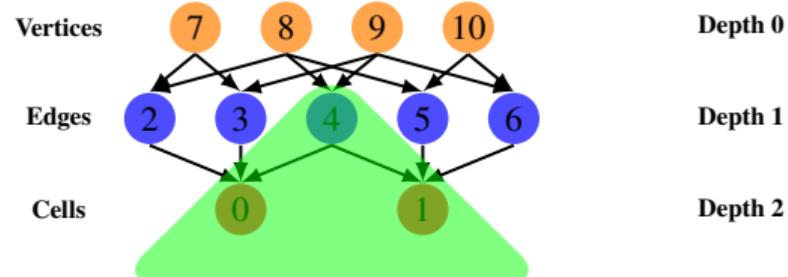
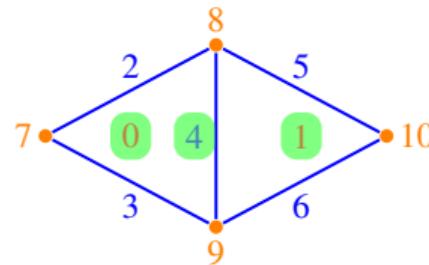


# Basic Operations

Meet

and augment with lattice operations.

$$\text{meet}(0, 1) = \{4\}$$

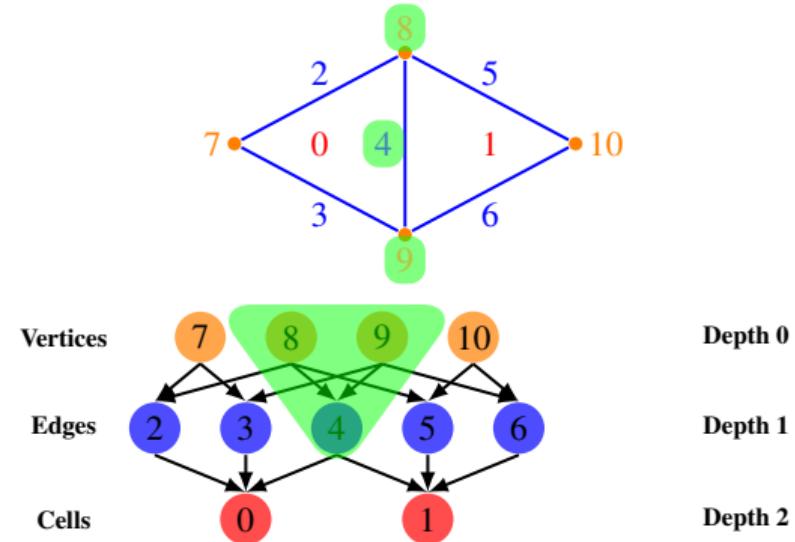


# Basic Operations

## Join

and augment with lattice operations.

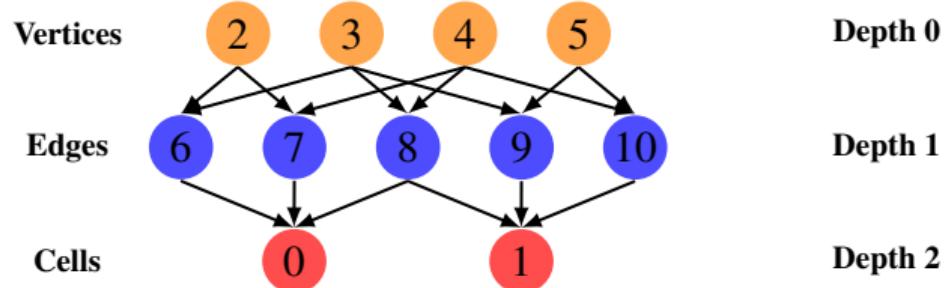
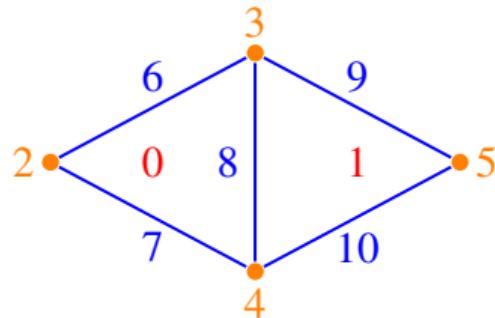
$$\text{join}(8, 9) = \{4\}$$



# Sample Meshes

Interpolated triangular mesh

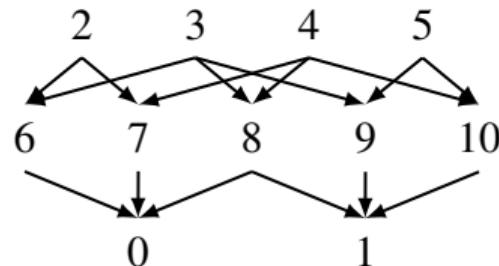
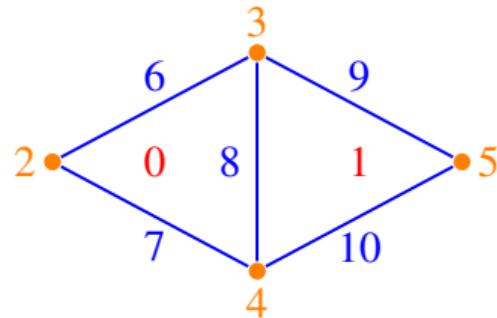
Starting with an interpolated mesh,



# Sample Meshes

Interpolated triangular mesh

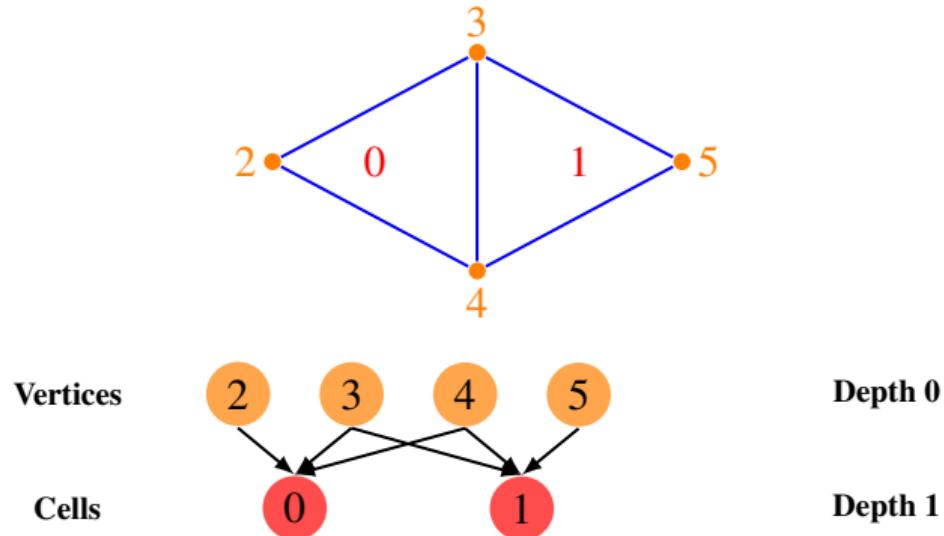
we can infer the depth in  $\mathcal{O}(N)$  (BFS),



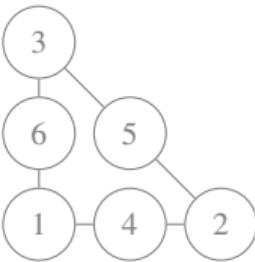
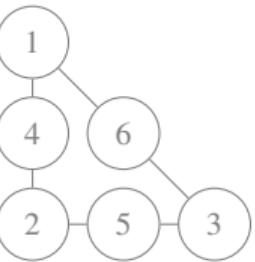
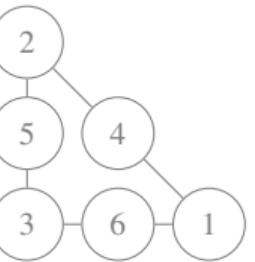
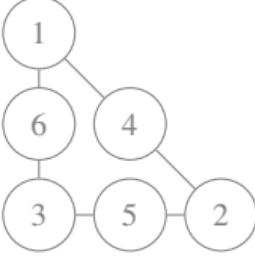
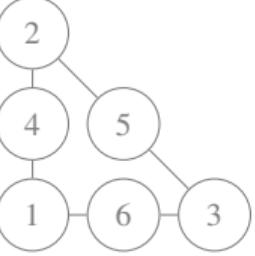
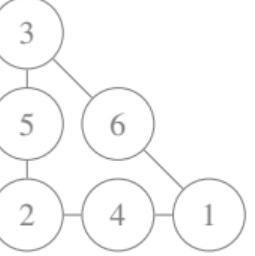
# Sample Meshes

Interpolated triangular mesh

and we can infer the intermediate levels in  $\mathcal{O}(N)$  (Join).



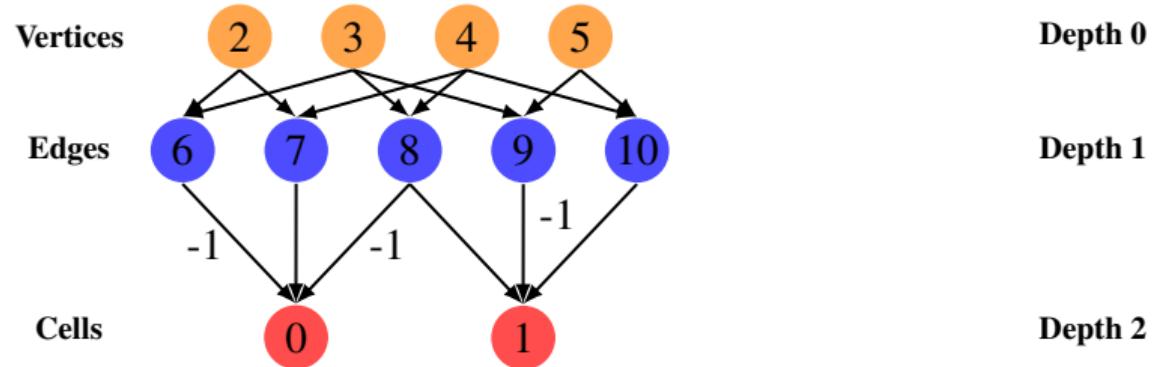
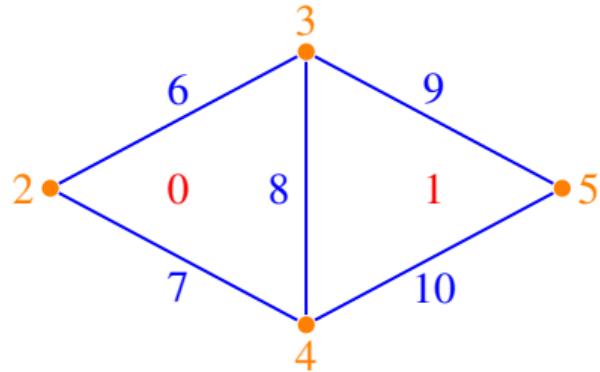
# Cell Orientation

Orientation	0	1	2
Arrangement			
Orientation	-1	-2	-3
Arrangement			

**Table:** The dihedral group  $D_3$  for the triangle

# Orientation

## Edge Decoration



# Outline

Meshes

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## Transformation

Production Rule:

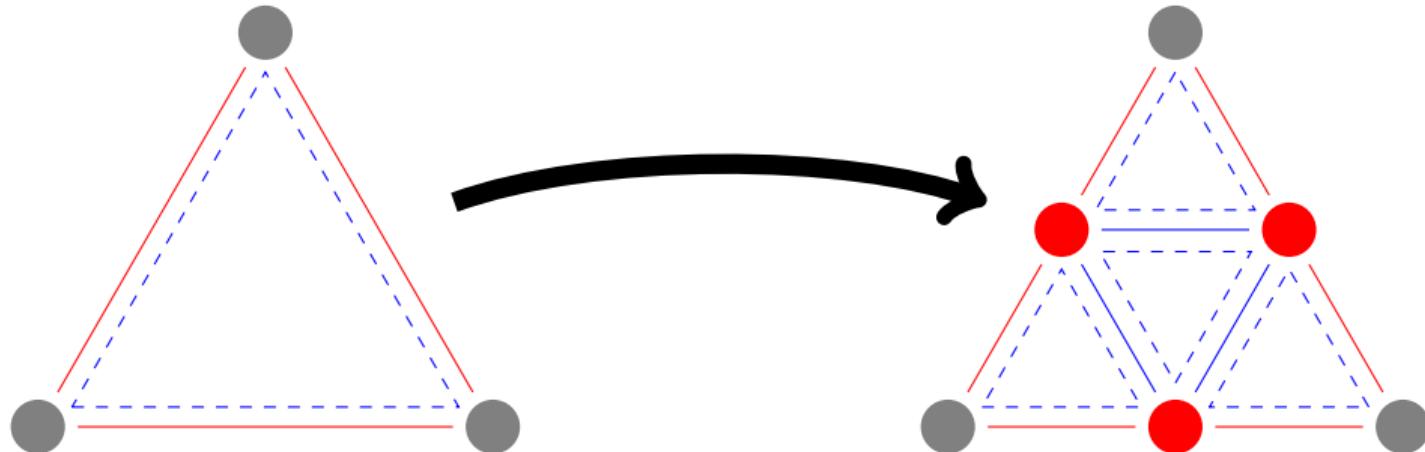
$$p \rightarrow \{(q_k, o_k)\}$$

or

$$q_k \in \text{child}(p)$$

# Example Transformation

## Refinement



## Transformation Conditions

Condition 1:

$$\text{cone}(\text{child}(p)) \in \text{child}(\text{cl}(p)),$$

Condition 2:

$$\text{parent}(\text{st}(q)) \in \text{st}(\text{parent}(q))$$

## Transformation Conditions

Condition 1:

$$\text{cone}(\text{child}(p)) \in \text{child}(\text{cl}(p)),$$

Condition 2:

$$|\text{parent}(q)| = 1$$

## Closure Complexity

Consider a point  $q'$  in the cone of  $q$ , so that

$$\begin{aligned} q' &\in \text{cone}(q) \\ &\in \text{child}(\text{cl}(p)) \end{aligned}$$

by Condition 1, meaning

$$\exists p' \in \text{cl}(p), q' \in \text{child}(p').$$

We may take the cone of each side and use Condition 1 again,

$$\begin{aligned} \text{cone}(q') &\in \text{cone}(\text{child}(p')) \\ &\in \text{child}(\text{cl}(p')) \\ &\in \text{child}(\text{cl}(p)) \end{aligned}$$

Thus

$$\text{cl}(\text{child}(p)) \in \text{child}(\text{cl}(p)).$$

## Support Complexity

Let  $p$  produce  $q$ , and consider a point  $q'$  in the star of  $q$ ,

$$q' \in \text{st}(q) \iff q \in \text{cl}(q')$$

Let  $q'$  be produced by a point  $p'$ ,

$$\begin{aligned} q' &\in \text{child}(p') \\ \text{cl}(q') &\in \text{child}(\text{cl}(p')) \\ q &\in \text{child}(\text{cl}(p')) \end{aligned}$$

Using Condition 2,

$$\begin{aligned} \text{parent}(q) &\in \text{cl}(p') \\ p &\in \text{cl}(p') \\ p' &\in \text{st}(p) \end{aligned}$$

and thus

$$\text{st}(\text{child}(p)) \in \text{child}(\text{st}(p)).$$

## Numbering

With unique parents,  
we can number everything *a priori*,  
in parallel.

## Output Sensitivity

To generate a closure in the output mesh,  
we need only the producing point closure.

To generate a star in the output mesh,  
we need only the producing point star.

# Outline

Meshes

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## Possible Proofs

- ▶ Transformation always produces a valid local CW-complex
- ▶ Transformation always produces a valid parallel CW-complex
- ▶ Transformation is output sensitive
- ▶ Transformation produces the intended output
- ▶ Parallel numbering with relaxed parent condition

# References I



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