Suppose we have a CFG \( G = (V, \Sigma, R, S) \) in Chomsky, NF. Set \( K = \mid V \mid, \ N = 2^K \).

Consider any \( x \in L(G) \) with \( \mid x \mid > N \) and take a parse tree \( T \) for \( x \) from \( S \).

By ChNF, \( T \) is a binary tree with all non-leaf nodes labeled by terminal chars.

Key Fact: By \( N \geq 2^K \), if \( \mid x \mid > N \) then \( T \) has a path with at least \( K+1 \) internal nodes. By Pigeonhole Principle some variable repeats along that path. Take a repeated variable \( D \) in the bottom \( K+1 \) nodes.
Redrawing central region of tree

I claim:

1. More done D to upper D. Kills the set: $x' = yuvwz = \text{ahhhbcchbd,} \text{d, bbbdd}$ nodes.
2. Repeat the lower D as the upper D for $i = 2$ times. Get $x^{(2)} = \text{ahhhbcchbd, bddbd,} \text{d, bbbdd}$.

Summary:

Original $x = yuvwz$

Got $x' = yu^0v^{(0)}w^0z = yuvz$

Got $x^{(2)} = yu^2v^{(2)}w^2z = yuvwz$ since $w = \epsilon$.

I show with $w = \epsilon$.

Always at least one of $u$ and $w$ must be nonzero.

Call this $x^{(0)}$.

Original $x = yuvwz$
Theorem: Given any context-free language $L$, there are numbers $K, N > 0$ such that for all $x \in L(G), \ |x| > N$, we can break $X = yu^i v w^j z$ s.t. $(\forall i \geq 0)$, the string $x^{(i)} = yu^i v w^j z$ belongs to $L$, and when $|uvw| \leq N$, and $u, w \neq \epsilon$. $G = (V, \Sigma, R, S)$

Proof: By $L$ being a CFL, $L$ has a grammar $G$ in GNF. Take $K = |V|$ and $N = 2^K$. The breakdown follows as above.

Contrapositive: Suppose for all $N > 0$ there exists $x \in L(G)$ with $|x| > N$ s.t. for all breakdowns $X = yu^i v w^j z$ where $|uvw| \leq N$ and $u, w \neq \epsilon$, there exists $i \geq 0$ such that $x^{(i)} = yu^i v w^j z \notin L$. Then $L$ is not a CFL.

This statement yields a "proof script" for proving that certain languages $L$ are not context-free.
Let any \( N > 0 \) be given. Take \( x = \_ \). Consider any possible breakdown \( x = yuvwz \) subject to \( |uw| \leq N \) and \( u \neq \emptyset \) and \( v \neq \emptyset \). Take \( i = \_ \). Then \( x(i) = yuvwz \) does not belong to \( L \) because \( \_ \).

Some \( N \) and the breakdown are arbitrary, \( L \) is not a CFL, by the CFL Pumping Lemma.

Example: \( L = \{ a^n b^n c^n : n \geq 1 \} \). Let \( N > 0 \) be given. Take \( x = a_N b_N c_N \). Then \( x \in L \). Visualize any possible breakdown \( x = yuvwz \) as:

\[ X: \]

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ad----a  b  b----b  c  c----c
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- Compass arms cannot be wider than \( N \).
- At least the \( u \) or \( v \) arm must be opened.

Hence the compass cannot keep all of the \( a, b, c \) in balance.

Hence \( x(0) = yuv \geq \) subtract at least one \( a, b, \) or \( c \).

\[ \_ \] \( x(0) \notin L \) because we cannot keep \#a = \#b = \#c in it. So \( L \) is not a CFL.