Thm. (Contrapositive)
Suppose for all $N > 0$, if $x \in L(G)$, $|x| > N$
s.t. for all breakdows $x = yuvwz$ where $|uvw| \leq N$
and $uw \neq \epsilon$,

$\exists i \geq 0$ s.t. $x^{(i)} = yuv^iz \notin L$,
then $L$ is NOT a CFL.

Template:
Let any $N > 0$ be given,

Take $X =$ ____________
Consider any possible breakdown $x = yuvwz$ subject to $|uvw| \leq N$
and $uw \neq \epsilon$

Take $i =$ ____________
Then $x^{(i)} = yu^iz \notin L$ because ____________

Since $N$ and the breakdown are arbitrary,
then $L$ is not a CFL by CFL Pumping Lemma.
Example 1. \( L = \{ a^i b^j c^k : i < j < k \} \)

Proof: Let any \( N > 0 \) be given.

Take \( X = a^{N+1} b^{N+2} c^{N} \).

Consider all possible breakdowns \( s.t. \ X = yuvwz \) with

\[ (6, c^N) \]

\( u/vw = a^n \), \( n \leq N, c^0 \)

\[ (6) \]

\( u/vw = \) at least \( c^0 \) has \( a^0 \)'s and \( c^0 \)'s collectively

\( (3) \)

\( u/vw \) has at least one \( c^0 \)

(1) "pumping up" to \( X(N+1) \)

\[ X(N+1) = yu^2 v w z = y \cdot a^{N+1} b^{N+2} c \neq L \]

for \( uvw = b^6 \)

"pumping down" to \( X^{(s)} = yvz \)

since \( u/vw \) has no \( a^0 \)'s or \( c^0 \)'s,

then \( X^{(s)} = yvz = a^N \cdot b^{N+1} c \neq L \) (CG) by \( uvw \neq \)

at least reduce \( 1 \) \( b^0 \)’s.

(2) for \( uvw = c^6 \),

similarly, "pumping down" to \( X^{(s)} = yvz \)

Either \( y, s \) could be \( 0 \), not both.

Hence, for \( z \geq 1 \), \( X^{(z)} = yu^z v w z \) has \( N + (z-1) \) \( a^0 \)'s

and \( N + (z-1) \cdot s \) \( b^0 \)’s, \( s \)

since \( y, s \geq 1 \)

take \( i = 3 \), \( X^{(3)} \) has at least \( N+2 \) \( a^0 \)'s and \( N+3 \) \( b^0 \)’s. \( \Rightarrow X^{(3)} \neq L \).
(3) "pumping down" to $X^{(0)} = y v z$,
then $X^{(0)}$ has 0 less than $\geq N+2$ many c's.

$\Rightarrow X^{(0)} \notin L$

Overall, all possible cases violate (Hi) $X^{(i)} \notin L$.

$\exists i, X^{(i)} \notin L$.

Therefore, I is not CFL.

Another way to consider all the breakdowns:

(1) $u = a^m$ for some $m > 0$,
then $X = y u^2 v w^2 z \notin L$, since $\#a(X^{(i)})$ is no less than

(2) $u = b^m$ for some $m > 0$,
then $X^{(0)} \notin L$

since $\#b$ of b's is not less than $\#a(X^{(i)})$.

(3) $u = c^m$, $m > 0$,
then $X^{(0)} \notin L$ since $\#b$ is not less than $\#c(X^{(i)})$.

(4) $u = z$ and replace u by w in the above three cases.
Example 2.

\[ L = \{ a^m b^n a^m b^n \} \]

Proof: Let any \( N > 0 \) be given, take \( X = a^n b^n a^n b^n = yuvwz \) with \( |uvw| \leq N \) and \( \text{uw} \neq \epsilon \).

**idea:** \( yuv \) must touch at least one of the four intervals, and at most two.

so for all possible cases:

"pumping down" to \( X^{(0)} = yuvz \) will reduce at least one of the \( a \)'s or \( b \)'s, thus \( X^{(0)} \notin L \).

\[ \Rightarrow \text{overall, } L \text{ is not a CFL.} \]

Q: What kind of model can recognize all those languages?
- Allow to change a char (regular, CFL, not CFL).
- Allow to move left.

Allowing only \( \bullet \) or \( \circ \) doesn't help.
Allowing both define a Turing Machine.
A Turing Machine (TM) or Deterministic TM (DTM),

* allowing to change chars one or more tapes,
* allowing tape head to move left (L) or stay (S)

besides moving right (R).

Def. A Turing Machine is a 7-tuple \( M = (Q, \Sigma, \Gamma, \delta, q_0, \sqcup, F) \)

where:
* \( Q \) is a finite set of states
* \( \Sigma \) is the finite input alphabet
* \( \Gamma \) is the tape alphabet, where \( B \in \Gamma \)
* \( \sqcup \) is the blank (\( \text{in text, or } 10 \text{ etc.} \))
* \( q_0 \) is the start state (\( q_0 \text{ in text} \))
* \( F \) is the set of desired final states

(in text, \( F = \{ q_{\text{acc}} \} \) where also W.L.O.G. there is a unique \( q_{\text{acc}} \))

\( S \subseteq Q \times \Gamma \times \Gamma \times \{ L, R, S \} \times Q \)

Diagram:

\[
\begin{array}{c}
\text{typical tuple} \\
(P, c, d, D, q) \\
\end{array}
\]
TM can decide languages like \( \{a^n b^n c^n : n \geq 1\} \) (not CFL)

Idea:

\[
\begin{array}{c}
\text{AAAABBBCCCC} \# \text{BB} \quad \downarrow \times \\
\text{XBBBBXXDCCC} \# \text{BB} \quad \downarrow \rightarrow \uparrow \\
\text{XXXA XXXX BXX XCC} \# \text{BB} \quad \uparrow
\end{array}
\]

Furthermore:

* M is deterministic if for all \( p \in Q \) and \( c \in \Gamma \), there is at most one tuple in \( S \) that begins \( (p, c/\ldots) \).

* M is "completed" if for all \( p \neq qaa, \text{ rej } \) and \( \emptyset \in C \Gamma \), there is a tuple beginning \( (p, c/\ldots) \) in the halting states.

Together \( \Rightarrow S \) is a function from \( (Q \setminus \{qaa, \text{ rej }\}, X \Gamma) \) to \( (\Gamma \times \{L, R, S\} \times Q) \).

Otherwise, if \( \exists \) any pair \( (q, c) \) with two or more tuples beginning \( (q, c/\ldots) \), then M is an NTM.