CSE 431/531: Algorithm Analysis and Design (Spring 2018)

Introduction and Syllabus

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo
1. Syllabus

2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times
Course Webpage (contains schedule, policies, homeworks and slides):
http://www.cse.buffalo.edu/~shil/courses/CSE531/

Please sign up the course on Piazza from course webpage polls, asking/answering questions.
Time and location:
- MoWeFr, 9:00-9:50am
- Talbert 107

Lecturer:
- Shi Li, shil@buffalo.edu
- Office hours: TBD

TAs
- TBD
You should know:

- **Mathematical Tools**
  - Mathematical inductions
  - Probabilities and random variables

- **Data Structures**
  - Stacks, queues, linked lists

- **Some Programming Experience**
  - E.g., C, C++, Java or Python
You Will Learn

- Classic algorithms for classic problems
  - Sorting
  - Shortest paths
  - Minimum spanning tree
- How to analyze algorithms
  - Correctness
  - Running time (efficiency)
  - Space requirement
- Meta techniques to design algorithms
  - Greedy algorithms
  - Divide and conquer
  - Dynamic programming
- NP-completeness
Tentative Schedule

- Introduction, 3 lectures
- Basic Graph Algorithms, 3 lectures
- Greedy Algorithms, 6 lectures (include recitation)
- Divide and Conquer, 6 lectures (include recitation)
- In-Class Exam #1, Mar 12, 2018
- Dynamic Programming, 6 lectures (include recitation)
- Linear Programming, 6 lectures (include recitation)
- In-Class Exam #2, Apr 18, 2018
- NP-Completeness 6 lectures (include recitation)
- Final Review, 1 lecture

Exercise problems will be posted before each recitation class
Textbook

Textbook (Highly Recommended):

- **Algorithm Design**, 1st Edition, by Jon Kleinberg and Eva Tardos

Other Reference Books

Highly recommended: read the correspondent sections from the textbook (or reference book) before classes

Slides will be posted online before class
Grading

- 40% for homeworks
  - 6 homeworks, 5 of which contain programming problems

- 60% for two in-class exams + final exam
  
  \[
  \max\{E_1 \times 5\% + F \times 25\%, \ E_1 \times 15\% + F \times 15\%\} \\
  + \max\{E_2 \times 5\% + F \times 25\%, \ E_2 \times 15\% + F \times 15\%\}
  \]

  \[E_1, E_2, F \in [0, 100]\]
For Homeworks, You Are Allowed to

- Use course materials (textbook, reference books, lecture notes, etc)
- Post questions on Piazza
- Ask me or TAs for hints
- Collaborate with classmates
  - Think about each problem for enough time before discussing
  - Must write down solutions on your own, in your own words
  - Write down names of students you collaborated with
For Homeworks, You Are Not Allowed to

- Use external resources
  - Can’t Google or ask questions online for solutions
  - Can’t read posted solutions from other algorithm courses
- Copy solutions from other students

If cheating is found, you will get an “F” for the course. The case will be reported to the department.
For Programming Problems

- Need to implement the algorithms by yourself
- Can not copy codes from others or the Internet

If cheating is found, you will get an “F” for the course. The case will be reported to the department.
Late policy

- You have one late credit
- Turn in a homework late for three days using the late credit
- No other late submissions will be accepted
Exams

- Closed-book
- Can bring one A4 handwritten sheet

If you are caught cheating in exams, you will get an “F” for the course. The case will be reported to the department.

Questions?
Outline

1 Syllabus

2 Introduction
   • What is an Algorithm?
   • Example: Insertion Sort
   • Analysis of Insertion Sort

3 Asymptotic Notations

4 Common Running times
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2. Introduction
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What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.

- Computational problem: specifies the input/output relationship.

- An algorithm solves a computational problem if it produces the correct output for any given input.
Examples

Greatest Common Divisor

**Input:** two integers \(a, b > 0\)

**Output:** the greatest common divisor of \(a\) and \(b\)

Example:

- Input: 210, 270
- Output: 30

- Algorithm: Euclidean algorithm
- \(\text{gcd}(270, 210) = \text{gcd}(210, 270 \mod 210) = \text{gcd}(210, 60)\)
- \((270, 210) \rightarrow (210, 60) \rightarrow (60, 30) \rightarrow (30, 0)\)
Examples

Sorting

**Input:** sequence of \( n \) numbers \((a_1, a_2, \cdots, a_n)\)

**Output:** a permutation \((a'_1, a'_2, \cdots, a'_n)\) of the input sequence such that \(a'_1 \leq a'_2 \leq \cdots \leq a'_n\)

Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59

- Algorithms: insertion sort, merge sort, quicksort, …
**Shortest Path**

**Input:** directed graph \( G = (V, E) \), \( s, t \in V \)

**Output:** a shortest path from \( s \) to \( t \) in \( G \)

- **Algorithm:** Dijkstra’s algorithm
Algorithm = Computer Program?

- Algorithm: “abstract”, can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: “concrete”, implementation of algorithm, associated with a particular programming language.
Pseudo-Code

Euclidean \((a, b)\)

1. \(\text{while } b > 0\)
2. \((a, b) \leftarrow (b, a \mod b)\)
3. \(\text{return } a\)

C++ program:

```cpp
int Euclidean(int a, int b){
    int c;
    while (b > 0){
        c = b;
        b = a % b;
        a = c;
    }
    return a;
}
```
Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g., GUI)
  - ...

Why is it important to study the running time (efficiency) of an algorithm?

1. feasible vs. infeasible
2. efficient algorithms: less engineering tricks needed, can use languages aiming for easy programming (e.g., python)
3. fundamental
4. it is fun!
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Sorting Problem

**Input:** sequence of \( n \) numbers \((a_1, a_2, \cdots, a_n)\)

**Output:** a permutation \((a'_1, a'_2, \cdots, a'_n)\) of the input sequence such that \( a'_1 \leq a'_2 \leq \cdots \leq a'_n \)

**Example:**

- Input: 53, 12, 35, 21, 59, 15
- Output: 12, 15, 21, 35, 53, 59
Insertion-Sort

At the end of $j$-th iteration, make the first $j$ numbers sorted.

- **iteration 1:** 53, 12, 35, 21, 59, 15
- **iteration 2:** 12, 53, 35, 21, 59, 15
- **iteration 3:** 12, 35, 53, 21, 59, 15
- **iteration 4:** 12, 21, 35, 53, 59, 15
- **iteration 5:** 12, 21, 35, 53, 59, 15
- **iteration 6:** 12, 15, 21, 35, 53, 59
Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59

### insertion-sort(A, n)

1. for $j \leftarrow 2$ to $n$
2. $\text{key} \leftarrow A[j]$
3. $i \leftarrow j - 1$
4. while $i > 0$ and $A[i] > \text{key}$
   - $A[i + 1] \leftarrow A[i]$
   - $i \leftarrow i - 1$
5. $A[i + 1] \leftarrow \text{key}$

- $j = 6$
- $\text{key} = 15$

12 15 21 35 53 59
↑
i
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Analysis of Insertion Sort

- Correctness
- Running time
Invariant: after iteration $j$ of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

- After $j = 1$: 53, 12, 35, 21, 59, 15
- After $j = 2$: 12, 53, 35, 21, 59, 15
- After $j = 3$: 12, 35, 53, 21, 59, 15
- After $j = 4$: 12, 21, 35, 53, 59, 15
- After $j = 5$: 12, 21, 35, 53, 59, 15
- After $j = 6$: 12, 15, 21, 35, 53, 59
Analyze Running Time of Insertion Sort

- Q: Size of input?
  - A: Running time as function of size
    - possible definition of size: # integers, total length of integers, # vertices in graph, # edges in graph

- Q: Which input?
  - A: Worst-case analysis:
    - Worst running time over all input instances of a given size

- Q: How fast is the computer?

- Q: Programming language?
  - A: Important idea: asymptotic analysis
    - Focus on growth of running-time as a function, not any particular value.
Asymptotic Analysis: $O$-notation

- Ignoring lower order terms
- Ignoring leading constant

1. $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
2. $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
3. $2^{n/3 + 100} + 100n^{100} \Rightarrow 2^{n/3 + 100} \Rightarrow 2^{n/3}$
4. $2^{n/3 + 100} + 100n^{100} = O(2^{n/3})$
Asymptotic Analysis: $O$-notation

- Ignoring lower order terms
- Ignoring leading constant

$O$-notation allows us to
- ignore architecture of computer
- ignore programming language
Asymptotic Analysis of Insertion Sort

**insertion-sort**($A, n$)

1. for $j \leftarrow 2$ to $n$
2. \hspace{1cm} $key \leftarrow A[j]$
3. \hspace{1cm} $i \leftarrow j - 1$
4. \hspace{1cm} while $i > 0$ and $A[i] > key$
5. \hspace{2cm} $A[i + 1] \leftarrow A[i]$
6. \hspace{2cm} $i \leftarrow i - 1$
7. \hspace{2cm} $A[i + 1] \leftarrow key$

- Worst-case running time for iteration $j$ in the outer loop? Answer: $O(j)$
- Total running time = $\sum_{j=2}^{n} O(j) = O(n^2)$ (informal)

Basic operations take $O(1)$ time: addition, subtraction, multiplication, etc.

Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough

Precision of real numbers? Try to avoid using real numbers in this course.

Can we do better than insertion sort asymptotically? Yes: merge sort, quicksort, heap sort, ...
Remember to sign up for Piazza.

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Asymptotically Positive Functions

Def. \( f : \mathbb{N} \rightarrow \mathbb{R} \) is an asymptotically positive function if:

- \( \exists n_0 > 0 \) such that \( \forall n > n_0 \) we have \( f(n) > 0 \)

In other words, \( f(n) \) is positive for large enough \( n \).

- \( n^2 - n - 30 \) \hspace{1cm} Yes
- \( 2^n - n^{20} \) \hspace{1cm} Yes
- \( 100n - n^2/10 + 50 \) \hspace{1cm} No

We only consider asymptotically positive functions.
**O-Notation** For a function $g(n)$, 

$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}$. 

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some $c$ and large enough $n$. 

![Graph showing the relationship between $f(n)$ and $cg(n)$, with a vertical line at $n_0$.](image)
**O-Notation**: Asymptotic Upper Bound

For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$  

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some $c$ and large enough $n$.

$3n^2 + 2n \in O(n^2 - 10n)$

**Proof.**

Let $c = 4$ and $n_0 = 10$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n) = -n^2 + 40n \leq 0.$$ 

$$3n^2 + 2n \leq c(n^2 - 10n)$$
**O-Notation**  For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}. $$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some $c$ and large enough $n$.

- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$
- $n^{100} \in O(2^n)$
- $n^3 \not\in O(10n^2)$

<table>
<thead>
<tr>
<th>Asymptotic Notations</th>
<th>$O$</th>
<th>$\Omega$</th>
<th>$\Theta$</th>
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<tbody>
<tr>
<td>Comparison Relations</td>
<td>$\leq$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Conventions

- We use \( f(n) = O(g(n)) \) to denote \( f(n) \in O(g(n)) \)
- \( 3n^2 + 2n = O(n^3 - 10n) \)
- \( 3n^2 + 2n = O(n^2 + 5n) \)
- \( 3n^2 + 2n = O(n^2) \)

“=” is asymmetric! Following statements are wrong:
- \( O(n^3 - 10n) = 3n^2 + 2n \)
- \( O(n^2 + 5n) = 3n^2 + 2n \)
- \( O(n^2) = 3n^2 + 2n \)
$\Omega$-Notation: Asymptotic Lower Bound

**$O$-Notation**  For a function $g(n)$,

$$O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.$$ 

**$\Omega$-Notation**  For a function $g(n)$,

$$\Omega(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0 \}.$$ 

- In other words, $f(n) \in \Omega(g(n))$ if $f(n) \geq cg(n)$ for some $c$ and large enough $n$. 
**Ω-Notation** For a function \( g(n) \),

\[ Ω(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0 \} . \]
Again, we use “=” instead of $\in$.

- $4n^2 = \Omega(n - 10)$
- $3n^2 - n + 10 = \Omega(n^2 - 20)$

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</table>

**Theorem**  
$f(n) = O(g(n))$ $\iff$ $g(n) = \Omega(f(n))$. 
**Θ-Notation**: Asymptotic Tight Bound

For a function $g(n)$,

$$\Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}.$$ 

- $f(n) = \Theta(g(n))$, then for large enough $n$, we have $f(n) \approx g(n)$. 

![Graph showing c1g(n), f(n), and c2g(n)]
**Θ-Notation**  For a function $g(n)$,
\[ Θ(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}. \]

- $3n^2 + 2n = Θ(n^2 - 20n)$
- $2^{n/3+100} = Θ(2^{n/3})$

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**Theorem**  $f(n) = Θ(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = Ω(g(n))$. 
## Exercise

For each pair of functions $f, g$ in the following table, indicate whether $f$ is $O$, $\Omega$ or $\Theta$ of $g$.

<table>
<thead>
<tr>
<th>$f$</th>
<th>$g$</th>
<th>$O$</th>
<th>$\Omega$</th>
<th>$\Theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^3 - 100n$</td>
<td>$5n^2 + 3n$</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$3n - 50$</td>
<td>$n^2 - 7n$</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>$n^2 - 100n$</td>
<td>$5n^2 + 30n$</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$\lg^{10} n$</td>
<td>$n^{0.1}$</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>$2^n$</td>
<td>$2^{n/2}$</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$\sqrt{n}$</td>
<td>$n\sin n$</td>
<td>No</td>
<td>No</td>
<td>No</td>
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Asymptotic Notations

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Trivial Facts on Comparison Relations

- $f \leq g \iff g \geq f$
- $f = g \iff f \leq g$ and $f \geq g$
- $f \leq g$ or $f \geq g$

Correct Analogies

- $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
- $f(n) = \Theta(g(n)) \iff f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Incorrect Analogy

- $f(n) = O(g(n))$ or $g(n) = O(f(n))$
Incorrect Analogy

- \( f(n) = O(g(n)) \) or \( g(n) = O(f(n)) \)

\[
\begin{align*}
  f(n) &= n^2 \\
  g(n) &= \begin{cases} 
    1 & \text{if } n \text{ is odd} \\
    2^n & \text{if } n \text{ is even}
  \end{cases}
\end{align*}
\]
Recall: informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \rightarrow 3n^2$
- ignoring leading constant: $3n^2 \rightarrow n^2$
- $3n^2 - 10n - 5 = O(n^2)$
- Indeed, $3n^2 - 10n - 5 = \Omega(n^2), 3n^2 - 10n - 5 = \Theta(n^2)$

- theoretically, nothing tells us to ignore lower order terms and leading constant.
- $3n^2 - 10n - 5 = O(5n^2 - 6n + 5)$ is correct, although weird.
- $3n^2 - 10n - 5 = O(n^2)$ is simpler.
$o$-Notation  For a function $g(n)$,
$$o(g(n)) = \{ \text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.$$  

$\omega$-Notation  For a function $g(n)$,
$$\omega(g(n)) = \{ \text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0 \}.$$  

Example:
- $3n^2 + 5n + 10 = o(n^2 \lg n)$.
- $3n^2 + 5n + 10 = \omega(n^2 / \lg n)$.
Asymptotic Notations

Comparison Relations

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Computing the sum of $n$ numbers

\[
\text{sum}(A, n)
\begin{align*}
1. & \quad S \leftarrow 0 \\
2. & \quad \text{for } i \leftarrow 1 \text{ to } n \\
3. & \quad \quad S \leftarrow S + A[i] \\
4. & \quad \text{return } S
\end{align*}
\]
\( O(n) \) (Linear) Running Time

- Merge two sorted arrays

\[
\begin{array}{cccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 \\
3 & 5 & 7 & 8 & 9 & 12 & 20 & 25 & 29 & 32 & 48
\end{array}
\]
\( O(n) \) (Linear) Running Time

\[
\text{merge}(B, C, n_1, n_2) \quad \text{\textbackslash\textbackslash B and C are sorted, with length } n_1 \text{ and } n_2
\]

1. \( A \leftarrow []; i \leftarrow 1; j \leftarrow 1 \)
2. while \( i \leq n_1 \) and \( j \leq n_2 \)
   3. if \( B[i] \leq C[j] \) then
      4. append \( B[i] \) to \( A \); \( i \leftarrow i + 1 \)
   5. else
      6. append \( C[j] \) to \( A \); \( j \leftarrow j + 1 \)
7. if \( i \leq n_1 \) then append \( B[i..n_1] \) to \( A \)
8. if \( j \leq n_2 \) then append \( C[j..n_2] \) to \( A \)
9. return \( A \)

Running time = \( O(n) \) where \( n = n_1 + n_2 \).
$O(n \log n)$ Running Time

merge-sort($A, n$)

1. if $n = 1$ then
2. return $A$
3. else
4. $B \leftarrow \text{merge-sort}\left(A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor\right)$
5. $C \leftarrow \text{merge-sort}\left(A[\lceil n/2 \rceil + 1..n], n - \lfloor n/2 \rfloor\right)$
6. return merge($B, C, \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor$)
$O(n \lg n)$ Running Time

- **Merge-Sort**

Each level takes running time $O(n)$
There are $O(\lg n)$ levels
Running time = $O(n \lg n)$
Closest Pair

**Input:** $n$ points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

**Output:** the pair of points that are closest
Closest Pair

**Input:** $n$ points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

**Output:** the pair of points that are closest

```plaintext
closest-pair(x, y, n)

1. $bestd \leftarrow \infty$
2. for $i \leftarrow 1$ to $n - 1$
   for $j \leftarrow i + 1$ to $n$
     $d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$
     if $d < bestd$ then
       $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$
3. return $(besti, bestj)$
```

Closest pair can be solved in $O(n \lg n)$ time!
Multiply two matrices of size $n \times n$

\[
\text{matrix-multiplication}(A, B, n)
\]

1. $C \leftarrow$ matrix of size $n \times n$, with all entries being 0
2. for $i \leftarrow 1$ to $n$
   3. for $j \leftarrow 1$ to $n$
      4. for $k \leftarrow 1$ to $n$
         5. $C[i, k] \leftarrow C[i, k] + A[i, j] \times B[j, k]$
6. return $C$
Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.

Independent set of size $k$

**Input:** graph $G = (V, E)$

**Output:** whether there is an independent set of size $k$
Independent Set of Size $k$

**Input:** graph $G = (V, E)$

**Output:** whether there is an independent set of size $k$

$$\text{independent-set}(G = (V, E))$$

1. for every set $S \subseteq V$ of size $k$
2. $b \leftarrow \text{true}$
3. for every $u, v \in S$
   - if $(u, v) \in E$ then $b \leftarrow \text{false}$
4. if $b$ return true
5. return false

Running time $= O\left(\frac{n^k}{k!} \times k^2\right) = O(n^k)$ (assume $k$ is a constant)
Beyond Polynomial Time: $O(2^n)$

**Maximum Independent Set Problem**

**Input:** graph $G = (V, E)$

**Output:** the maximum independent set of $G$

**max-independent-set($G = (V, E)$)**

1. $R \leftarrow \emptyset$
2. for every set $S \subseteq V$
3. \hspace{0.5cm} $b \leftarrow \text{true}$
4. \hspace{1cm} for every $u, v \in S$
5. \hspace{1.5cm} if $(u, v) \in E$ then $b \leftarrow \text{false}$
6. \hspace{1cm} if $b$ and $|S| > |R|$ then $R \leftarrow S$
7. return $R$

Running time = $O(2^n n^2)$. 
Beyond Polynomial Time: $O(n!)$

**Hamiltonian Cycle Problem**

**Input:** a graph with $n$ vertices

**Output:** a cycle that visits each node exactly once, or say no such cycle exists
Beyond Polynomial Time: \( n! \)

**Hamiltonian** \((G = (V, E))\)

1. for every permutation \((p_1, p_2, \cdots, p_n)\) of \(V\)
2. \(b \leftarrow \text{true}\)
3. for \(i \leftarrow 1\) to \(n - 1\)
   - if \((p_i, p_{i+1}) \notin E\) then \(b \leftarrow \text{false}\)
4. if \((p_n, p_1) \notin E\) then \(b \leftarrow \text{false}\)
5. if \(b\) then return \((p_1, p_2, \cdots, p_n)\)
6. return “No Hamiltonian Cycle”

Running time = \(O(n! \times n)\)
$O(lg \ n)$ (Logarithmic) Running Time

- Binary search
  - Input: sorted array $A$ of size $n$, an integer $t$;
  - Output: whether $t$ appears in $A$.

E.g, search 35 in the following array:

3 8 10 25 29 37 38 42 46 52 59 61 63 75 79
Binary search

- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$.

$\text{binary-search}(A, n, t)$

1. $i \leftarrow 1, j \leftarrow n$
2. while $i \leq j$ do
3. \hspace{1em} $k \leftarrow \lfloor (i + j)/2 \rfloor$
4. \hspace{1em} if $A[k] = t$ return true
5. \hspace{1em} if $A[k] < t$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
6. return false

Running time $= O(\lg n)$
Compare the Orders

- Sort the functions from smallest to largest asymptotically
  \( n^{\sqrt{n}}, \ lg\ n, \ n, \ n^2, \ n \ lg \ n, \ n!, \ 2^n, \ e^n, \ lg(n!), \ n^n \)
- \( f < g \) stands for \( f = o(g) \), \( f = g \) stands for \( f = \Theta(g) \)
- \( lg\ n < n^{\sqrt{n}} \)
- \( lg\ n < n < n^{\sqrt{n}} \)
- \( lg\ n < n < n^2 < n^{\sqrt{n}} \)
- \( lg\ n < n < n \ lg \ n < n^2 < n^{\sqrt{n}} < n! \)
- \( lg\ n < n < n \ lg \ n < n^2 < n^{\sqrt{n}} < 2^n < n! \)
- \( lg\ n < n < n \ lg \ n = lg(n!) < n^2 < n^{\sqrt{n}} < 2^n < e^n < n! \)
- \( lg\ n < n < n \ lg \ n = lg(n!) < n^2 < n^{\sqrt{n}} < 2^n < e^n < n! < n^n \)
Terminologies

When we talk about upper bounds:

- Logarithmic time: $O(\lg n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant $k$
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$

When we talk about lower bounds:

- Super-linear time: $\omega(n)$
- Super-quadratic time: $\omega(n^2)$
- Super-polynomial time: $\bigcap_{k>0} \omega(n^k)$
Goal of Algorithm Design

- Design algorithms to minimize the order of the running time.
- Using asymptotic analysis allows us to ignore the leading constants and lower order terms.
- Makes our life much easier! (E.g., the leading constant depends on the implementation, compiler and computer architecture of computer.)
Q: Does ignoring the leading constant cause any issues?

- e.g., how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

- Sometimes yes
- However, when $n$ is big enough, $1000n < 0.1n^2$
- For “natural” algorithms, constants are not so big!
- So, for reasonable $n$, algorithm with lower order running time beats algorithm with higher order running time.