CSE 431/531: Algorithm Analysis and Design (Spring 2018)

Introduction and Syllabus

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo
Outline

1 Syllabus

2 Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3 Asymptotic Notations

4 Common Running times
Course Webpage (contains schedule, policies, homeworks and slides):
http://www.cse.buffalo.edu/~shil/courses/CSE531/

Please sign up the course on Piazza from course webpage polls, asking/answering questions.
Time and location:
  - MoWeFr, 9:00-9:50am
  - Talbert 107

Lecturer:
  - Shi Li, shil@buffalo.edu
  - Office hours: TBD

TAs
  - TBD
You should know:
You should know:

- **Mathematical Tools**
  - Mathematical inductions
  - Probabilities and random variables
You should know:

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  - Mathematical inductions
  - Probabilities and random variables

- **Data Structures**
  - Stacks, queues, linked lists
You should know:

- **Mathematical Tools**
  - Mathematical inductions
  - Probabilities and random variables

- **Data Structures**
  - Stacks, queues, linked lists

- **Some Programming Experience**
  - E.g., C, C++, Java or Python
You Will Learn

- Classic algorithms for classic problems
  - Sorting
  - Shortest paths
  - Minimum spanning tree

- How to analyze algorithms
  - Correctness
  - Running time (efficiency)
  - Space requirement

- Meta techniques to design algorithms
  - Greedy algorithms
  - Divide and conquer
  - Dynamic programming
  - Linear Programming

- NP-completeness
You Will Learn

- Classic algorithms for classic problems
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- NP-completeness
Tentative Schedule

- Introduction, 3 lectures
- Basic Graph Algorithms, 3 lectures
- Greedy Algorithms, 6 lectures (include recitation)
- Divide and Conquer, 6 lectures (include recitation)
- In-Class Exam #1, Mar 12, 2018
- Dynamic Programming, 6 lectures (include recitation)
- Linear Programming, 6 lectures (include recitation)
- In-Class Exam #2, Apr 18, 2018
- NP-Completeness, 6 lectures (include recitation)
- Final Review, 1 lecture

Exercise problems will be posted before each recitation class
Textbook (Highly Recommended):

- **Algorithm Design**, 1st Edition, by Jon Kleinberg and Eva Tardos

Other Reference Books

Reading Before Classes

- Highly recommended: read the correspondent sections from the textbook (or reference book) before classes
- Slides will be posted online before class
40% for homeworks

- 6 homeworks, 5 of which contain programming problems
Grading

- 40% for homeworks
  - 6 homeworks, 5 of which contain programming problems

- 60% for two in-class exams + final exam

\[
\max\{E_1 \times 5\% + F \times 25\%, \, E_1 \times 15\% + F \times 15\%\} \\
+ \max\{E_2 \times 5\% + F \times 25\%, \, E_2 \times 15\% + F \times 15\%\}
\]

\[E_1, \, E_2, \, F \in [0, 100]\]
For Homeworks, You Are Allowed to

- Use course materials (textbook, reference books, lecture notes, etc)
- Post questions on Piazza
- Ask me or TAs for hints
- Collaborate with classmates
  - Think about each problem for enough time before discussing
  - Must write down solutions on your own, in your own words
  - Write down names of students you collaborated with
For Homeworks, You Are Not Allowed to

- Use external resources
  - Can’t Google or ask questions online for solutions
  - Can’t read posted solutions from other algorithm courses
- Copy solutions from other students

If cheating is found, you will get an "F" for the course. The case will be reported to the department.
For Homeworks, You Are **Not Allowed to**

- Use external resources
  - Can’t Google or ask questions online for solutions
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For Programming Problems

- Need to implement the algorithms by your self
- Can not copy codes from others or the Internet
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If cheating is found, you will get an “F” for the course. The case will be reported to the department.
Late policy

- You have one late credit
- Turn in a homework late for three days using the late credit
- No other late submissions will be accepted
Exams

- Closed-book
- Can bring one A4 handwritten sheet

If you are caught cheating in exams, you will get an “F” for the course. The case will be reported to the department.

Questions?
Exams

- Closed-book
- Can bring one A4 handwritten sheet

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Exams

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Questions?
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   • What is an Algorithm?
   • Example: Insertion Sort
   • Analysis of Insertion Sort

3 Asymptotic Notations

4 Common Running times
Outline

1. Syllabus

2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times
What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.
What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.

- Computational problem: specifies the input/output relationship.

- An algorithm **solves** a computational problem if it produces the correct output for any given input.
## Greatest Common Divisor

**Input:** two integers $a, b > 0$

**Output:** the greatest common divisor of $a$ and $b$

**Examples**

Example:

Input: 210, 270

Output: 30

Algorithm: Euclidean algorithm

$\text{gcd}(270, 210) = \text{gcd}(210, 270 \mod 210) = \text{gcd}(210, 60) \rightarrow (210, 60) \rightarrow (60, 30) \rightarrow (30, 0)$
Examples

Greatest Common Divisor

**Input:** two integers $a, b > 0$

**Output:** the greatest common divisor of $a$ and $b$

Example:
- Input: 210, 270
- Output: 30

Algorithm: Euclidean algorithm

\[
gcd(270, 210) = gcd(210, 270 \mod 210) = gcd(210, 60)
\]

\[
(270, 210) \rightarrow (210, 60) \rightarrow (60, 30) \rightarrow (30, 0)
\]
Examples

**Greatest Common Divisor**

**Input:** two integers \(a, b > 0\)

**Output:** the greatest common divisor of \(a\) and \(b\)

**Example:**
- Input: 210, 270
- Output: 30

- Algorithm: Euclidean algorithm
## Examples

### Greatest Common Divisor

**Input:** two integers $a, b > 0$

**Output:** the greatest common divisor of $a$ and $b$

### Example:

- **Input:** 210, 270
- **Output:** 30

- **Algorithm:** Euclidean algorithm
- $\gcd(270, 210) = \gcd(210, 270 \mod 210) = \gcd(210, 60)$
Examples

**Greatest Common Divisor**

**Input:** two integers \(a, b > 0\)

**Output:** the greatest common divisor of \(a\) and \(b\)

**Example:**

- **Input:** 210, 270
- **Output:** 30

- **Algorithm:** Euclidean algorithm
- \(\text{gcd}(270, 210) = \text{gcd}(210, 270 \mod 210) = \text{gcd}(210, 60)\)
- \((270, 210) \rightarrow (210, 60) \rightarrow (60, 30) \rightarrow (30, 0)\)
### Examples

**Sorting**

**Input:** sequence of $n$ numbers $(a_1, a_2, \cdots, a_n)$

**Output:** a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$
Examples

Sorting

**Input:** sequence of $n$ numbers $(a_1, a_2, \cdots, a_n)$

**Output:** a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59
Examples

**Sorting**

**Input:** sequence of $n$ numbers $(a_1, a_2, \cdots, a_n)$

**Output:** a permutation $(a_1', a_2', \cdots, a_n')$ of the input sequence such that $a_1' \leq a_2' \leq \cdots \leq a_n'$

**Example:**
- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59

**Algorithms:** insertion sort, merge sort, quicksort, ...
**Shortest Path**

**Input:** directed graph $G = (V, E)$, $s, t \in V$

**Output:** a shortest path from $s$ to $t$ in $G$
**Examples**

**Shortest Path**

**Input:** directed graph \( G = (V, E) \), \( s, t \in V \)

**Output:** a shortest path from \( s \) to \( t \) in \( G \)

![Graph Diagram]

Algorithm: Dijkstra's algorithm
**Examples**

**Shortest Path**

**Input:** directed graph $G = (V, E)$, $s, t \in V$

**Output:** a shortest path from $s$ to $t$ in $G$

![Diagram of a directed graph with nodes s, 3, and t and edges with weights 1, 4, 5, 16, 1, 2, 4, 10, and 3]
Examples

Shortest Path

**Input:** directed graph $G = (V, E)$, $s, t \in V$

**Output:** a shortest path from $s$ to $t$ in $G$

- Algorithm: Dijkstra’s algorithm
Algorithm = Computer Program?

- Algorithm: “abstract”, can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: “concrete”, implementation of algorithm, associated with a particular programming language
Pseudo-Code:

Euclidean\((a, b)\)

1. while \(b > 0\)
2. \((a, b) \leftarrow (b, a \mod b)\)
3. return \(a\)

C++ program:

```cpp
int Euclidean(int a, int b){
    int c;
    while (b > 0){
        c = b;
        b = a % b;
        a = c;
    }
    return a;
}
```
Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)

Why is it important to study the running time (efficiency) of an algorithm?

1. Feasible vs. infeasible
2. Efficient algorithms: less engineering tricks needed, can use languages aiming for easy programming (e.g., python)
3. Fundamental
4. It is fun!
Main focus: correctness, running time (efficiency)
Sometimes: memory usage
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Sometimes: memory usage
Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g., GUI)
  - ...

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**Input:** sequence of \( n \) numbers \((a_1, a_2, \cdots, a_n)\)

**Output:** a permutation \((a'_1, a'_2, \cdots, a'_n)\) of the input sequence such that \(a'_1 \leq a'_2 \leq \cdots \leq a'_n\)

**Example:**

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59
At the end of $j$-th iteration, make the first $j$ numbers sorted.

iteration 1: 53, 12, 35, 21, 59, 15
iteration 2: 12, 53, 35, 21, 59, 15
iteration 3: 12, 35, 53, 21, 59, 15
iteration 4: 12, 21, 35, 53, 59, 15
iteration 5: 12, 21, 35, 53, 59, 15
iteration 6: 12, 15, 21, 35, 53, 59
Example:

- Input: 53, 12, 35, 21, 59, 15
- Output: 12, 15, 21, 35, 53, 59

**insertion-sort**(*A, n*)

1. for *j* ← 2 to *n*
2. \[ key \leftarrow A[j] \]
3. \[ i \leftarrow j - 1 \]
4. while *i* > 0 and *A[i]* > *key*
5. \[ A[i + 1] \leftarrow A[i] \]
6. \[ i \leftarrow i - 1 \]
7. \[ A[i + 1] \leftarrow key \]
Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59

**insertion-sort(A, n)**

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   - \( A[i + 1] \leftarrow A[i] \)
   - \( i \leftarrow i - 1 \)
5. \( A[i + 1] \leftarrow key \)

- \( j = 6 \)
- \( key = 15 \)

12 21 35 53 59 15

\[\uparrow\]

\[i\]
Example:
- Input: 53, 12, 35, 21, 59, 15
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$\uparrow$

$i$
Example:
- Input: 53, 12, 35, 21, 59, 15
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insertion-sort(A, n)

1. for j ← 2 to n
2.   key ← A[j]
3.   i ← j - 1
4.   while i > 0 and A[i] > key
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- $12 \ 21 \ 35 \ 53 \ 53 \ 59$
  \[\uparrow\]
  \[i\]

28/74
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- Input: 53, 12, 35, 21, 59, 15
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insertion-sort($A, n$)

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   $key \leftarrow A[j]$
3.   $i \leftarrow j - 1$
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2. \hspace{1em} key $\leftarrow A[j]$
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4. while $i > 0$ and $A[i] > key$
5. \hspace{1em} $A[i + 1] \leftarrow A[i]$
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- Input: 53, 12, 35, 21, 59, 15
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insertion-sort(A, n)
1. for j ← 2 to n
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Analysis of Insertion Sort

- Correctness
- Running time
Invariant: after iteration $j$ of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

- after $j = 1$: 53, 12, 35, 21, 59, 15
- after $j = 2$: 12, 53, 35, 21, 59, 15
- after $j = 3$: 12, 35, 53, 21, 59, 15
- after $j = 4$: 12, 21, 35, 53, 59, 15
- after $j = 5$: 12, 21, 35, 53, 59, 15
- after $j = 6$: 12, 15, 21, 35, 53, 59
Analyze Running Time of Insertion Sort

Q: Size of input?
Q: Size of input?
A: Running time as function of size
Q: Size of input?
A: Running time as function of size
possible definition of size: # integers, total length of integers, # vertices in graph, # edges in graph
Analyze Running Time of Insertion Sort

- Q: Size of input?
- A: Running time as function of size
  
  possible definition of size: # integers, total length of integers, # vertices in graph, # edges in graph

- Q: Which input?

Worst-case analysis:
Worst running time over all input instances of a given size

Q: How fast is the computer?

Q: Programming language?

Important idea: asymptotic analysis
Focus on growth of running-time as a function, not any particular value.
Q: Size of input?

A: Running time as function of size

possible definition of size: # integers, total length of integers, # vertices in graph, # edges in graph

Q: Which input?

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Worst running time over all input instances of a given size
Analyze Running Time of Insertion Sort

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  possible definition of size: # integers, total length of integers, # vertices in graph, # edges in graph

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possible definition of size: # integers, total length of integers, # vertices in graph, # edges in graph

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- possible definition of size: # integers, total length of integers, # vertices in graph, # edges in graph

- Q: Which input?
- A: Worst-case analysis:
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- Q: How fast is the computer?
- Q: Programming language?

- A: Important idea: asymptotic analysis
  - Focus on growth of running-time as a function, not any particular value.
Asymptotic Analysis: $O$-notation

- Ignoring lower order terms
- Ignoring leading constant
Asymptotic Analysis: $O$-notation

- Ignoring lower order terms
- Ignoring leading constant

$3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
Asymptotic Analysis: $O$-notation

- Ignoring lower order terms
- Ignoring leading constant

\[ 3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3 \]

\[ 3n^3 + 2n^2 - 18n + 1028 = O(n^3) \]
Asymptotic Analysis: $O$-notation

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $2^{n/3 + 100} + 100n^{100} \Rightarrow 2^{n/3 + 100} \Rightarrow 2^{n/3}$
Asymptotic Analysis: $O$-notation

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $\frac{2^{n/3+100}}{} + 100n^{100} \Rightarrow \frac{2^{n/3+100}}{} \Rightarrow \frac{2^{n/3}}{}$
- $\frac{2^{n/3+100}}{} + 100n^{100} = O(2^{n/3})$
Asymptotic Analysis: $O$-notation

- Ignoring lower order terms
- Ignoring leading constant

$O$-notation allows us to

- ignore architecture of computer
- ignore programming language
Asymptotic Analysis of Insertion Sort

**insertion-sort**(*A, n*)

1. for *j* ← 2 to *n*
2. \( \text{key} \leftarrow A[j] \)
3. \( i \leftarrow j - 1 \)
4. while *i* > 0 and \( A[i] > \text{key} \)
5. \( A[i + 1] \leftarrow A[i] \)
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7. \( A[i + 1] \leftarrow \text{key} \)

Worst-case running time for iteration *j* in the outer loop?
Answer: \( O(j) \)

Total running time = \( \sum_{j=2}^{n} O(j) = O(n^2) \) (informal)
Asymptotic Analysis of Insertion Sort

```
insertion-sort(A, n)

1 for j ← 2 to n
2   key ← A[j]
3   i ← j - 1
4   while i > 0 and A[i] > key
5     A[i + 1] ← A[i]
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```

- Worst-case running time for iteration $j$ in the outer loop?
Asymptotic Analysis of Insertion Sort

<table>
<thead>
<tr>
<th>insertion-sort(A, n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. for j ← 2 to n</td>
</tr>
<tr>
<td>2. key ← A[j]</td>
</tr>
<tr>
<td>3. i ← j - 1</td>
</tr>
<tr>
<td>4. while i &gt; 0 and A[i] &gt; key</td>
</tr>
<tr>
<td>6. i ← i - 1</td>
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<tr>
<td>7. A[i + 1] ← key</td>
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- Worst-case running time for iteration $j$ in the outer loop?
  
  Answer: $O(j)$
Asymptotic Analysis of Insertion Sort

**insertion-sort**(*A*, *n*)

1. for *j* ← 2 to *n*
2. \( key \leftarrow A[j] \)
3. \( i \leftarrow j - 1 \)
4. while \( i > 0 \) and \( A[i] > key \)
5. \( A[i + 1] \leftarrow A[i] \)
6. \( i \leftarrow i - 1 \)
7. \( A[i + 1] \leftarrow key \)

- Worst-case running time for iteration *j* in the outer loop?
  - Answer: \( O(j) \)
- Total running time = \( \sum_{j=2}^{n} O(j) = O(n^2) \) (informal)

Basic operations take $O(1)$ time: addition, subtraction, multiplication, etc.

Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough

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Can we do better than insertion sort asymptotically?

Yes: merge sort, quicksort, heap sort, ...
Remember to sign up for Piazza.

Questions?
Outline

1. Syllabus

2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times
Def. $f : \mathbb{N} \rightarrow \mathbb{R}$ is an asymptotically positive function if:

• $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$
Asymptotically Positive Functions

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We only consider asymptotically positive functions.
**O-Notation**: For a function $g(n)$,

$$O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.$$
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**Proof.**

Let \( c = 4 \) and \( n_0 = 10 \), for every \( n > n_0 = 50 \), we have,
\[
3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)
\]
\[
= -n^2 + 40n \leq 0.
\]
\[
3n^2 + 2n \leq c(n^2 - 10n)
\]
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### Asymptotic Notations

<table>
<thead>
<tr>
<th>Comparison Relations</th>
<th>( O )</th>
<th>( \Omega )</th>
<th>( \Theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \leq )</td>
<td></td>
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We use \( f(n) = O(g(n)) \) to denote \( f(n) \in O(g(n)) \)
We use “$f(n) = O(g(n))$” to denote “$f(n) \in O(g(n))$”

- $3n^2 + 2n = O(n^3 - 10n)$
- $3n^2 + 2n = O(n^2 + 5n)$
- $3n^2 + 2n = O(n^2)$
Conventions

- We use “$f(n) = O(g(n))$” to denote “$f(n) \in O(g(n))$”
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“$=$” is asymmetric! Following statements are wrong:
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“\( = \)” is asymmetric! Following statements are wrong:
- \( O(n^3 - 10n) = 3n^2 + 2n \)
- \( O(n^2 + 5n) = 3n^2 + 2n \)
- \( O(n^2) = 3n^2 + 2n \)
$\Omega$-Notation: Asymptotic Lower Bound

**O-Notation**  For a function $g(n)$,

$$O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.$$

**Ω-Notation**  For a function $g(n)$,

$$\Omega(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0 \}.$$
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Again, we use “=” instead of \( \in \).

- \( 4n^2 = \Omega(n - 10) \)
- \( 3n^2 - n + 10 = \Omega(n^2 - 20) \)
Again, we use “=” instead of ∈.

- \(4n^2 = \Omega(n - 10)\)
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**Theorem** \[ f(n) = O(g(n)) \iff g(n) = \Omega(f(n)). \]
\( \Theta \)-Notation: Asymptotic Tight Bound

**\( \Theta \)-Notation**  For a function \( g(n) \),

\[
\Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}. 
\]
**Θ-Notation**: Asymptotic Tight Bound

**Θ-Notation**  For a function $g(n)$,

$$
\Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}.
$$

- $f(n) = \Theta(g(n))$, then for large enough $n$, we have “$f(n) \approx g(n)$”.
**Θ-Notation** For a function $g(n)$,

$$
\Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } \\
\quad c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}.
$$

- $f(n) = \Theta(g(n))$, then for large enough $n$, we have “$f(n) \approx g(n)$”.

![Graph showing the relationship between $c_1g(n)$, $f(n)$, and $c_2g(n)$ over $n$.]
**Θ-Notation: Asymptotic Tight Bound**

**Θ-Notation** For a function $g(n)$,

$\Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that} \}
\begin{align*}
c_1 g(n) &\leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}\}.
\end{align*}$

- $3n^2 + 2n = \Theta(n^2 - 20n)$
**Θ-Notation** For a function \( g(n) \),

\[
\Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}.
\]

- \( 3n^2 + 2n = \Theta(n^2 - 20n) \)
- \( 2^{n/3} + 100 = \Theta(2^{n/3}) \)
For a function $g(n)$,
\[ \Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}. \]

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**Θ-Notation**  For a function $g(n)$,

$$\Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}.$$ 

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**Theorem**  $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. 
Exercise

For each pair of functions $f, g$ in the following table, indicate whether $f$ is $O, \Omega$ or $\Theta$ of $g$.

<table>
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<th></th>
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<tr>
<td>1</td>
<td>$n^3 - 100n$</td>
<td>$5n^2 + 3n$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$3n - 50$</td>
<td>$n^2 - 7n$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$n^2 - 100n$</td>
<td>$5n^2 + 30n$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$\lg_{10} n$</td>
<td>$n^{0.1}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$2^n$</td>
<td>$2^{n/2}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$\sqrt{n}$</td>
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<td>$n^2 - 7n$</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>$n^2 - 100n$</td>
<td>$5n^2 + 30n$</td>
<td>Yes</td>
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</tr>
<tr>
<td>$\lg_{10} n$</td>
<td>$n^{0.1}$</td>
<td></td>
<td></td>
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</tr>
<tr>
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</table>
**Exercise**

For each pair of functions $f, g$ in the following table, indicate whether $f$ is $O, \Omega$ or $\Theta$ of $g$.

<table>
<thead>
<tr>
<th>$f$</th>
<th>$g$</th>
<th>$O$</th>
<th>$\Omega$</th>
<th>$\Theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^3 - 100n$</td>
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### Asymptotic Notations

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### Trivial Facts on Comparison Relations

- $f \leq g \iff g \geq f$
- $f = g \iff f \leq g$ and $f \geq g$
- $f \leq g$ or $f \geq g$

Correct Analogies

- $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
- $f(n) = \Theta(g(n)) \iff f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Incorrect Analogy

- $f(n) = O(g(n))$ or $g(n) = O(f(n))$
Asymptotic Notations

<table>
<thead>
<tr>
<th>O</th>
<th>Ω</th>
<th>Θ</th>
</tr>
</thead>
</table>

Comparison Relations

| ≤ | ≥ | = |

Trivial Facts on Comparison Relations

- $f \leq g \iff g \geq f$
- $f = g \iff f \leq g \text{ and } f \geq g$
- $f \leq g \text{ or } f \geq g$

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- $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
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Asymptotic Notations

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Trivial Facts on Comparison Relations

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Correct Analogies

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Incorrect Analogy

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Incorrect Analogy

- \( f(n) = O(g(n)) \) or \( g(n) = O(f(n)) \)

\[
\begin{align*}
  f(n) &= n^2 \\
  g(n) &= \begin{cases} 
    1 & \text{if } n \text{ is odd} \\
    2^n & \text{if } n \text{ is even}
  \end{cases}
\end{align*}
\]
Recall: informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \rightarrow 3n^2$
- ignoring leading constant: $3n^2 \rightarrow n^2$
Recall: informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \rightarrow 3n^2$
- ignoring leading constant: $3n^2 \rightarrow n^2$
- $3n^2 - 10n - 5 = O(n^2)$

Indeed, $3n^2 - 10n - 5 = \Omega(n^2)$, $3n^2 - 10n - 5 = \Theta(n^2)$ is correct, although weird. $3n^2 - 10n - 5 = O(5n^2 - 6n + 5)$ is simpler.
Recall: informal way to define $O$-notation

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- $3n^2 - 10n - 5 = O(n^2)$ is simpler.
\( o\) and \( \omega\)-Notations

**o-Notation**  For a function \( g(n) \),

\[
o(g(n)) = \{ \text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.\]

**\( \omega\)-Notation**  For a function \( g(n) \),

\[
\omega(g(n)) = \{ \text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0 \}.\]

Example:

- \( 3n^2 + 5n + 10 = o(n^2 \lg n) \).
- \( 3n^2 + 5n + 10 = \omega(n^2 / \lg n) \).
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<thead>
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Questions?
1. Syllabus

2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times
Computing the sum of $n$ numbers

$$\text{sum}(A, n)$$

1. $S \leftarrow 0$
2. for $i \leftarrow 1$ to $n$
3. \hspace{1em} $S \leftarrow S + A[i]$
4. return $S$
Merge two sorted arrays

3 8 12 20 32 48

5 7 9 25 29

\(O(n)\) (Linear) Running Time
Merge two sorted arrays

\[
\begin{array}{cccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 \\
\end{array}
\]
Merge two sorted arrays

3 8 12 20 32 48
5 7 9 25 29
3
$O(n)$ (Linear) Running Time

- Merge two sorted arrays

<table>
<thead>
<tr>
<th>3</th>
<th>8</th>
<th>12</th>
<th>20</th>
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<td>9</td>
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<td>29</td>
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<tr>
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$O(n)$ (Linear) Running Time

- Merge two sorted arrays

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5 7 9 25 29
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- Merge two sorted arrays

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3 8 12 20 32 48
5 7 9 25 29
3 5
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Merge two sorted arrays

3 8 12 20 32 48

5 7 9 25 29

3 5 7
**Merge two sorted arrays**

- First array: 3 8 12 20 32 48
- Second array: 5 7 9 25 29
- Result: 3 5 7
Merge two sorted arrays

\[
\begin{array}{cccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 \\
3 & 5 & 7 & 8 \\
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- Merge two sorted arrays

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3 8 12 20 32 48
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Merge two sorted arrays

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**$O(n)$ (Linear) Running Time**

- Merge two sorted arrays

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\begin{array}{ccccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 \\
3 & 5 & 7 & 8 & 9 & 12 & 20 & 25 & 29 & 32 & 48
\end{array}
\]
**O(n)** (Linear) Running Time

merge\((B, C, n_1, n_2)\) \quad B and C are sorted, with length \(n_1\) and \(n_2\)

1. \(A \leftarrow []; i \leftarrow 1; j \leftarrow 1\)
2. while \(i \leq n_1\) and \(j \leq n_2\)
3. \hspace{1em} if \((B[i] \leq C[j])\) then
4. \hspace{2em} append \(B[i]\) to \(A\); \(i \leftarrow i + 1\)
5. \hspace{1em} else
6. \hspace{2em} append \(C[j]\) to \(A\); \(j \leftarrow j + 1\)
7. \hspace{1em} if \(i \leq n_1\) then append \(B[i..n_1]\) to \(A\)
8. \hspace{1em} if \(j \leq n_2\) then append \(C[j..n_2]\) to \(A\)
9. return \(A\)

Running time = \(O(n)\) where \(n = n_1 + n_2\).
merge($B, C, n_1, n_2$) \ \ \ B and C are sorted, with length $n_1$ and $n_2$

1. $A \leftarrow []; i \leftarrow 1; j \leftarrow 1$
2. while $i \leq n_1$ and $j \leq n_2$
3. \ \ if ($B[i] \leq C[j]$) then
4. \ \ \ \ append $B[i]$ to $A; i \leftarrow i + 1$
5. \ \ else
6. \ \ \ \ append $C[j]$ to $A; j \leftarrow j + 1$
7. \ \ if $i \leq n_1$ then append $B[i..n_1]$ to $A$
8. \ \ if $j \leq n_2$ then append $C[j..n_2]$ to $A$
9. return $A$

Running time $= O(n)$ where $n = n_1 + n_2$. 
merge-sort($A, n$)

1. if $n = 1$ then
2. return $A$
3. else
4. $B \leftarrow$ merge-sort($A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor$)
5. $C \leftarrow$ merge-sort($A[\lceil n/2 \rceil + 1..n], n - \lfloor n/2 \rfloor$)
6. return merge($B, C, \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor$)
Merge-Sort

$$O(n \log n)$$ Running Time

- Each level takes running time $$O(n)$$
- There are $$O(\log n)$$ levels
- Running time = $$O(n \log n)$$
$O(n \lg n)$ Running Time

- Merge-Sort

- Each level takes running time $O(n)$
Merge-Sort

Each level takes running time \( O(n) \)

There are \( O(\log n) \) levels
**$O(n \lg n)$ Running Time**

- **Merge-Sort**

![Diagram of Merge-Sort](image)

- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$
Closest Pair

**Input:** \( n \) points in plane: \((x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\)

**Output:** the pair of points that are closest
Closest Pair

**Input:** $n$ points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

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Closest Pair

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closest-pair\((x, y, n)\)

1. \( \text{bestd} \leftarrow \infty \)
2. for \( i \leftarrow 1 \) to \( n - 1 \)
3. for \( j \leftarrow i + 1 \) to \( n \)
4. \( d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2} \)
5. if \( d < \text{bestd} \) then
6. \( \text{besti} \leftarrow i, \text{bestj} \leftarrow j, \text{bestd} \leftarrow d \)
7. return \((\text{besti}, \text{bestj})\)
Closest Pair

Input: $n$ points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest

closest-pair($x, y, n$)

1. $bestd \leftarrow \infty$
2. for $i \leftarrow 1$ to $n - 1$
3. for $j \leftarrow i + 1$ to $n$
4. $d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$
5. if $d < bestd$ then
6. $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$
7. return $(besti, bestj)$

Closest pair can be solved in $O(n \lg n)$ time!
Multiply two matrices of size $n \times n$

\begin{algorithm}
\textbf{matrix-multiplication}($A, B, n$)
\begin{algorithmic}[1]
\State $C \leftarrow$ matrix of size $n \times n$, with all entries being 0
\For{$i \leftarrow 1$ to $n$}
\For{$j \leftarrow 1$ to $n$}
\For{$k \leftarrow 1$ to $n$}
\State $C[i, k] \leftarrow C[i, k] + A[i, j] \times B[j, k]$
\EndFor
\EndFor
\EndFor
\State return $C$
\end{algorithmic}
\end{algorithm}
Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$. 
Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$. 

$O(n^k)$ Running Time for Integer $k \geq 4$
\( O(n^k) \) Running Time for Integer \( k \geq 4 \)

**Def.** An independent set of a graph \( G = (V, E) \) is a subset \( S \subseteq V \) of vertices such that for every \( u, v \in S \), we have \((u, v) \notin E\).
Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.

Independent set of size $k$

**Input:** graph $G = (V, E)$

**Output:** whether there is an independent set of size $k$
Independent Set of Size $k$

**Input:** graph $G = (V, E)$

**Output:** whether there is an independent set of size $k$

**independent-set($G = (V, E)$)**

1. for every set $S \subseteq V$ of size $k$
2. $b \leftarrow$ true
3. for every $u, v \in S$
4. if $(u, v) \in E$ then $b \leftarrow$ false
5. if $b$ return true
6. return false

Running time $= O\left(\frac{n^k}{k!} \times k^2\right) = O(n^k)$ (assume $k$ is a constant)
Beyond Polynomial Time: $O(2^n)$

**Maximum Independent Set Problem**

**Input:** graph $G = (V, E)$

**Output:** the maximum independent set of $G$

```
max-independent-set(G = (V, E))
```

1. $R \leftarrow \emptyset$
2. for every set $S \subseteq V$
3. \quad $b \leftarrow \text{true}$
4. \quad for every $u, v \in S$
5. \quad \quad if $(u, v) \in E$ then $b \leftarrow \text{false}$
6. \quad if $b$ and $|S| > |R|$ then $R \leftarrow S$
7. return $R$

Running time = $O(2^n n^2)$. 
Hamiltonian Cycle Problem

**Input:** a graph with $n$ vertices

**Output:** a cycle that visits each node exactly once, or say no such cycle exists
Hamiltonian Cycle Problem

**Input:** a graph with $n$ vertices

**Output:** a cycle that visits each node exactly once, or say no such cycle exists.
Beyond Polynomial Time: \( n! \)

```plaintext
Hamiltonian(G = (V, E))

1. for every permutation \((p_1, p_2, \cdots, p_n)\) of V
2. \( b \leftarrow \text{true} \)
3. for \( i \leftarrow 1 \) to \( n - 1 \)
4. \hspace{1em} if \((p_i, p_{i+1}) \notin E\) then \( b \leftarrow \text{false} \)
5. \hspace{1em} if \((p_n, p_1) \notin E\) then \( b \leftarrow \text{false} \)
6. \hspace{1em} if \( b \) then return \((p_1, p_2, \cdots, p_n)\)
7. return “No Hamiltonian Cycle”

Running time = \( O(n! \times n) \)
```
$O(\lg n)$ (Logarithmic) Running Time

**Input:** sorted array $A$ of size $n$, an integer $t$;  
**Output:** whether $t$ appears in $A$. 

E.g., search 35 in the following array:
Binary search

- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$. 

E.g., search 35 in the following array:
Binary search

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E.g, search 35 in the following array:

<p>| | | | | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
<td>10</td>
<td>25</td>
<td>29</td>
<td>37</td>
<td>38</td>
<td>42</td>
<td>46</td>
<td>52</td>
<td>59</td>
<td>61</td>
<td>63</td>
<td>75</td>
<td>79</td>
<td></td>
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</tbody>
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- Binary search
  - Input: sorted array $A$ of size $n$, an integer $t$;
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- E.g., search 35 in the following array:

```
3  8 10 25 29 37 38 42 46 52 59 61 63 75 79
```

42 > 35
Binary search

- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$.

E.g, search 35 in the following array:

\[
\begin{array}{cccccccccccccccc}
3 & 8 & 10 & 25 & 29 & 37 & 38 & 42 & 46 & 52 & 59 & 61 & 63 & 75 & 79 \\
\end{array}
\]
Binary search

- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$.

E.g, search 35 in the following array:
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  - Input: sorted array $A$ of size $n$, an integer $t$;
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- E.g, search 35 in the following array:

```
3  8  10  25  29  37  38  42  46  52  59  61  63  75  79
```
$O(\lg n)$ (Logarithmic) Running Time

Binary search

- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$.

\[
\text{binary-search}(A, n, t)
1. \quad i \leftarrow 1, j \leftarrow n
2. \quad \text{while } i \leq j \text{ do}
3. \quad \quad k \leftarrow \lfloor (i + j)/2 \rfloor
4. \quad \quad \text{if } A[k] = t \text{ return true}
5. \quad \quad \text{if } A[k] < t \text{ then } j \leftarrow k - 1 \text{ else } i \leftarrow k + 1
6. \quad \text{return false}
\]
**Binary search**

- **Input:** sorted array $A$ of size $n$, an integer $t$;
- **Output:** whether $t$ appears in $A$.

**binary-search($A$, $n$, $t$)**

1. $i \leftarrow 1$, $j \leftarrow n$
2. while $i \leq j$ do
3. \hspace{1em} $k \leftarrow \lfloor (i + j)/2 \rfloor$
4. \hspace{1em} if $A[k] = t$ return true
5. \hspace{1em} if $A[k] < t$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
6. return false

**Running time** = $O(\lg n)$
Compare the Orders

- Sort the functions from smallest to largest asymptotically

\[ n^{\sqrt{n}}, \ lg\ n, \ n, \ n^2, \ n\ lg\ n, \ n!, \ 2^n, \ e^n, \ lg(n!), \ n^n \]
Compare the Orders

- Sort the functions from smallest to largest asymptotically:
  \( n^{\sqrt{n}}, \ lg \ n, \ n, \ n^2, \ n \ lg \ n, \ n!, \ 2^n, \ e^n, \ lg(n!), \ n^n \)
  
- \( f < g \) stands for \( f = o(g) \), \( f = g \) stands for \( f = \Theta(g) \).
  
- \( \lg n < n^{\sqrt{n}} \)
Compare the Orders

- Sort the functions from smallest to largest asymptotically
  \(n\sqrt{n}, \ lg\ n, \ n, \ n^2, \ n\ lg\ n, \ n!, \ 2^n, \ e^n, \ lg(n!), \ n^n\)
- \(f < g\) stands for \(f = o(g)\), \(f = g\) stands for \(f = \Theta(g)\)!
- \(\lg\ n < n\sqrt{n}\)
- \(\lg\ n < n < n\sqrt{n}\)
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- \( \lg\ n < n < n^{\sqrt{n}} \)
- \( \lg\ n < n < n^2 < n^{\sqrt{n}} \)
- \( \lg\ n < n < n \ lg\ n < n^2 < n^{\sqrt{n}} \)
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- \( lg\ n < n < n^{\sqrt{n}} \)
- \( lg\ n < n < n^2 < n^{\sqrt{n}} \)
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- \( lg\ n < n < n \ lg\ n < n^2 < n^{\sqrt{n}} < n! \)
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Compare the Orders

- Sort the functions from smallest to largest asymptotically:
  \( n^{\sqrt{n}} \), \( \lg n \), \( n \), \( n^2 \), \( n \lg n \), \( n! \), \( 2^n \), \( e^n \), \( \lg(n!) \), \( n^n \)

- \( f < g \) stands for \( f = o(g) \), \( f = g \) stands for \( f = \Theta(g) \! \)

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- \( \lg n < n < n^2 < n^{\sqrt{n}} \)
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- \( \lg n < n < n \lg n = \lg(n!) < n^2 < n^{\sqrt{n}} < 2^n < e^n < n! \)
Compare the Orders

- Sort the functions from smallest to largest asymptotically:
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When we talk about upper bounds:

- Logarithmic time: $O(\lg n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant $k$
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$
Terminologies

When we talk about upper bounds:

- Logarithmic time: $O(\log n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant $k$
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$

When we talk about lower bounds:

- Super-linear time: $\omega(n)$
- Super-quadratic time: $\omega(n^2)$
- Super-polynomial time: $\bigcap_{k>0} \omega(n^k)$
Goal of Algorithm Design

- Design algorithms to minimize the order of the running time.

Using asymptotic analysis allows us to ignore the leading constants and lower order terms, making our life much easier! (E.g., the leading constant depends on the implementation, compiler and computer architecture of computer.)
Goal of Algorithm Design

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- Using asymptotic analysis allows us to ignore the leading constants and lower order terms

- Makes our life much easier! (E.g., the leading constant depends on the implementation, compiler and computer architecture of computer.)
Q: Does ignoring the leading constant cause any issues?

- e.g., how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?
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- Sometimes yes
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- e.g., how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

- Sometimes yes
- However, when $n$ is big enough, $1000n < 0.1n^2$
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A:
- Sometimes yes
- However, when $n$ is big enough, $1000n < 0.1n^2$
- For “natural” algorithms, constants are not so big!
Q: Does ignoring the leading constant cause any issues?

- e.g., how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

- Sometimes yes
- However, when $n$ is big enough, $1000n < 0.1n^2$
- For “natural” algorithms, constants are not so big!
- So, for reasonable $n$, algorithm with lower order running time beats algorithm with higher order running time.