19.1 Ellipsoid Method

Consider the following example.

**Example 1:**
LP relaxation for travelling salesman problem
Given: a metric $d$ over $V$,
Goal: find a shortest tour to visit all vertices in $V$

Let’s say $x_{\{u,v\}} \in \{0,1\}$ denotes whether $\{u,v\}$ is used in the tour.
So, the integer programming for the problem will be the following:

$$\begin{align*}
\text{min} & \quad \sum_{\{u,v\}} x_{\{u,v\}} \cdot d(u,v) \\
\text{subject to} & \quad \sum_{u \neq v} x_{\{u,v\}} = 2 \quad (\forall v \in V) \\
& \quad \sum_{\{u,v\}:|\{u,v\} \cap S|=1} x_{\{u,v\}} \geq 2 \quad (\forall S \subseteq V, S \neq \emptyset) \\
& \quad x_{\{u,v\}} \in \{0,1\} \quad (\forall u, v \in V)
\end{align*}$$

For LP relaxation, we will only change the last condition into $x_{\{u,v\}} \geq 0 \quad (\forall u, v \in V)$.

**Separation Oracle O:**
Given $x$, O will
- accept $x$ if $x$ satisfies all the constraints,
- reject $x$ if $x$ does not satisfy all the constraints and return a constraint that $x$ violates.

**Efficient seperation oracle for LP relaxation of TSP:**
Using max-flow-min-cut theorem (MFMC), fix $s \in V$, enumerate $t \in V \setminus \{s\}$, check if we can send 2 units flow from $s$ to $t$, in the network with capacities $\{x_{u,v}\}_{u,v}$.
If for some $t$, we cannot send 2 units flow from $s$ to $t$, then by MFMC theorem, we can find a cut $(s,v \setminus S), s \in S, t \notin S$ such that

$$\sum_{\{u,v\}:|\{u,v\} \cap S|=1} x_{\{u,v\}} < 2$$
otherwise, all constraints are satisfied.

\[
y^t_{(v,u)} + y^t_{(u,v)} \leq x_{(u,v)} \quad (\forall u, v)
\]

\[
\sum_{u \neq v} y^t_{(v,u)} \quad (\forall v \notin \{s, t\}) - \text{Flow Conservation-}
\]

\[
\sum_{u \neq s} y^t_{(s,u)} = 2 \quad (\forall s)
\]

\[
y^t_{(u,s)} = 0 \quad (\forall u)
\]

**Example 2:**

Multicut Problem

Given: \( G = (V, E) \), cost \( \{c_e\}_{e \in E} \),

\((s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)\) pairs of vertices in \( V \).

Goal: find a set \( E' \) of edges such that \( s_i \) and \( t_i \) are disconnected in \( (V, E \setminus E') \); \( (\forall i \in [k]) \),

minimize \( \sum_{e \in E'} c_e \).

Now, let \( x_e \) denote whether \( e \in E' \) or not, in other words whether we are removing edge \( e \) to obtain the multicut. Thus, the LP will be the following:

\[
\begin{align*}
\min \sum_{e \in E} c_e \\
\sum_{e \in P} x_e & \geq 1 \quad (\forall \text{ path } P \text{ connecting } s_i \text{ to } t_i, \text{ for some } i) \\
x_e & \geq 0 \quad (\forall e \in E)
\end{align*}
\]

**Efficient separation oracle for LP relaxation of multicut:**

For every \( i \), find the shortest path from \( s_i \) to \( t_i \), using \( \{x_e\} \) as costs.

If distance from \( s_i \) to \( t_i \) is less than 1, then return \( P=\text{shortest path} \), otherwise accept \( x \).

Now, this is where we are

Given: separation oracle \( O \), \( c \in \mathbb{R}^n \),

Goal: \( \min C^T x \) such that \( x \) is accepted by \( O \).

\* Ellipsoid is obtained from a ball by scaling and rotation.
Algorithm 1 Ellipsoid Method

\[ P \leftarrow \text{ellipsoid containing all feasible solutions} \]

\[ \text{while } P \text{ is "not small enough" do} \]
\[ \hat{x} \leftarrow \text{center of } P \]
\[ \text{query O whether } \hat{x} \text{ is feasible} \]
\[ \text{if Yes then} \]
\[ P' \leftarrow \{x \in P : CTx \leq CT\hat{x}\} \]
\[ x^* \leftarrow \hat{x} \]
\[ \text{else} \]
\[ \text{let } ax \geq b \text{ be the constraint that } \hat{x} \text{ violated returned by O} \]
\[ P' \leftarrow \{x \in P : ax \geq b\} \]
\[ P \leftarrow \text{small ellipsoid containing } P' \]

Lemma 19.1 We can guarantee that volume of \( P \) is at most \( (1 - \frac{1}{2^n}) \) times volume of \( P \) in the previous iteration.

Number of iterations = \( O(n.lg \frac{\text{initial volume of } P}{\text{minimal possible value}}) \)