8.1 Set Coverage Problem

Problem:

\[ \min \sum_{i=1}^{m} y_i \quad \text{s.t.} \]

\[ \sum_{i,j \in S_i} y_i \geq 1, \forall j \in [n] \]

\[ y_i \geq 0, \forall i \in [m] \]

\( \tilde{y}_i = 0 \) for every \( i \in [m] \)

**Algorithm 1** Algorithm for Set Cover LP

1. \textbf{do}
2. Choose \( i^* \) according to the following distribution:
3. \( Pr[i^* = i] = \frac{y_i}{\sum_{i'=1}^{m} y_{i'}} \)
4. let \( \tilde{y}_{i^*} \leftarrow \tilde{y}_{i^*} + 1 \)
5. \textbf{until} all elements are covered by \( \{S_i : \tilde{y}_i \geq 1\} \)

**Lemma 8.1**

\[ \mathbb{E}[\tilde{x}_j] \geq (1 - \frac{1}{e})x_j \]

\[ \Rightarrow \mathbb{E}[\sum_{j=1}^{n} \tilde{x}_j] \geq (1 - \frac{1}{e} \sum_{j=1}^{n} x_j) \]

**Proof:**

\[ \mathbb{E}[\tilde{x}_j] = 1 - (1 - \frac{\sum_{i:j \in S_i} y_i}{\sum_{i=1}^{m} y_i})^k \geq 1 - (1 - \frac{\sum_{i:j \in S_i} y_i}{k})^k \]

\[ \geq 1 - (1 - \frac{x_j}{k})^{k\ln(n^2)} \]
\[ \geq 1 - e^{x_j \ln(n^2)} \]

Need:
\[ \frac{1 - e^{x_j \ln(n^2)}}{x_j} \geq 1 - \frac{1}{e^{\ln(n^2)}} = 1 - \frac{1}{n^2} \]

### 8.2 \( f - approximation \) for Set Coverage Problem

**Define**: \( f = \max_j |\{i : j \in S_i\}| \)

**Goal**: obtain an \( f \)-approximation for set cover

This problem is similar to vertex cover problem:

- Given \( G = (V, E) \), \( V \) : sets, \( E \) : elements
  - Choose a minimum - size set \( U \subseteq V \) s.t. \( \forall (u, v) \in E, \{u, v\} \cap U \neq \emptyset \)

LP rounding algorithm that gives \( f - approximation \) for set-cover problem.

\[
\tilde{y}_i = \begin{cases} 
1 & \text{if } y_i \geq \frac{1}{f} \\
0 & \text{otherwise}
\end{cases}
\]

return \( \{S_i : \tilde{y}_i = 1\} \)

\[ \sum_{i=1}^{m} \tilde{y}_i \leq f \sum_{i=1}^{m} y_i \]

This is also works for weighted problem:

\[ \sum_{i=1}^{m} w_i \tilde{y}_i \leq f \sum_{i=1}^{m} w_i y_i \]